

# AOSS Mathematical Reference Sheets

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September 8th, 2008

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# 1 Trigonometry

## Reciprocal properties

$$\begin{aligned}\sin x &= \frac{1}{\csc x} & \cos x &= \frac{1}{\sec x} \\ \tan x &= \frac{\sin x}{\cos x} & \cot x &= \frac{1}{\tan x}\end{aligned}$$

## Pythagorean identities

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ \sec^2 x - \tan^2 x &= 1 \\ \csc^2 x - \cot^2 x &= 1\end{aligned}$$

## Periodicity and Reflection

$$\begin{aligned}\sin\left(\frac{\pi}{2} - x\right) &= \cos x & \sin(-x) &= -\sin(x) \\ \cos\left(\frac{\pi}{2} - x\right) &= \sin x & \cos(-x) &= \cos(x) \\ \tan\left(\frac{\pi}{2} - x\right) &= \cot x & \tan(-x) &= -\tan(x)\end{aligned}$$

## Complex notation

$$\begin{aligned}\sin x &= \frac{1}{2i} (e^{ix} - e^{-ix}) \\ \cos x &= \frac{1}{2} (e^{ix} + e^{-ix})\end{aligned}$$

## Sum-Difference Formulas

$$\begin{aligned}\sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \\ \cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\ \tan(a \pm b) &= \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}\end{aligned}$$

## Double Angle Formulas

$$\begin{aligned}\sin(2x) &= 2 \sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2 \cos^2 x - 1 \\ &= 1 - 2 \sin^2 x \\ \tan(2x) &= \frac{2 \tan x}{1 - \tan^2 x}\end{aligned}$$

## Sum-Product Formulas

$$\begin{aligned}\sin a + \sin b &= 2 \sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right) \\ \sin a - \sin b &= 2 \cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right) \\ \cos a + \cos b &= 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right) \\ \cos a - \cos b &= -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)\end{aligned}$$

## Product-Sum Formulas

$$\begin{aligned}\sin a \sin b &= \frac{1}{2} [\cos(a-b) - \cos(a+b)] \\ \cos a \cos b &= \frac{1}{2} [\cos(a-b) + \cos(a+b)] \\ \sin a \cos b &= \frac{1}{2} [\sin(a+b) + \sin(a-b)] \\ \cos a \sin b &= \frac{1}{2} [\sin(a+b) - \sin(a-b)]\end{aligned}$$

# 2 Hyperbolic Trigonometry

## Definition

$$\begin{aligned}\sinh x &= \frac{1}{2} (e^x - e^{-x}) \\ \cosh x &= \frac{1}{2} (e^x + e^{-x}) \\ \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}}\end{aligned}$$

## Inverse Definition

$$\begin{aligned}\operatorname{arsinh}(z) &= \ln(z + \sqrt{z^2 + 1}) \\ \operatorname{arcosh}(z) &= \ln(z + \sqrt{z^2 - 1}) \\ \operatorname{arctanh}(z) &= \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)\end{aligned}$$

## Reflection

$$\begin{aligned}\sinh(-x) &= -\sinh(x) \\ \cosh(-x) &= \cosh(x) \\ \tanh(-x) &= -\tanh(x)\end{aligned}$$

## Relation with Elliptic Trigonometry

$$\begin{aligned}\sinh x &= -i \sin(ix) \\ \cosh x &= \cos(ix) \\ \tanh x &= -i \tan(ix)\end{aligned}$$

## Pythagorean Identities

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= 1 \\ \operatorname{sech}^2 x + \tanh^2 x &= 1\end{aligned}$$

## Double Angle Formulas

$$\begin{aligned}\sinh(2x) &= 2 \sinh x \cosh x \\ \cosh(2x) &= \cosh^2 x + \sinh^2 x \\ \tanh(2x) &= \frac{2 \tanh x}{1 + \tanh^2 x}\end{aligned}$$

# 3 Single and Multivariable Calculus

## First Fundamental Theorem of Calculus

Suppose  $f$  continuous and real valued on  $[a, b]$ , then

$$F(x) = \int_a^x f(t)dt \iff \frac{dF}{dx}(x) = f(x).$$

## Second Fundamental Theorem of Calculus

Suppose  $f$  continuous and real valued on  $[a, b]$  and let  $F$  satisfy  $f(x) = \frac{dF}{dx}(x)$ , then

$$\int_a^b f(x)dx = F(b) - F(a).$$

## Chain Rule

If  $f = f(y(x))$  then

$$\frac{df}{dx} = \frac{df}{dy} \frac{dy}{dx}$$

## Liebniz (Product) Rule

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

## Basic Integration by Parts

$$\int f dg = fg - \int gdf$$

## Multivariable Chain Rule

If  $f = f(x_1(t), \dots, x_n(t))$  then

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \dots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}$$

## Jacobian

$$J(x_1, \dots, x_n) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

## Integration by Parts

Suppose  $\Omega$  is an open bounded subset of  $\mathbf{R}^n$  with a piecewise smooth boundary  $\partial\Omega$ . If  $u(x_1, \dots, x_n)$  and  $v(x_1, \dots, x_n)$  are continuously differentiable on  $\Omega \cup \partial\Omega$  then

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = \int_{\partial\Omega} uv \hat{n}_i - \int_{\Omega} u \frac{\partial v}{\partial x_i} dx.$$

## Change of Variables (1D)

Suppose  $f$  is a continuous function of  $y$  on some domain  $S = [a, b]$  and  $T = [\phi(a), \phi(b)]$  is the range  $[a, b]$  under the differentiable mapping  $y = \phi(x)$ . Then

$$\int_T f(\phi(x)) \frac{d\phi}{dx} dx = \int_S f(y) dy.$$

## Change of Variables (2D)

Suppose  $f$  is a continuous function of  $(x, y)$  on some domain  $S$  and  $T = \Psi(S)$  is an orientation-preserving diffeomorphism. Then

$$\int_T f(x, y) dx dy = \int_S f[x(u, v), y(u, v)] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv,$$

where

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix},$$

is the Jacobian in 2D.

## Implicit Function Theorem

Given  $F_i(u_1, \dots, u_n, x_1, \dots, x_n) = 0$  for  $i = 1, \dots, n$ , then if the Jacobian satisfies

$$|JF(u_1, \dots, u_n)| = \left| \frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)} \right| \neq 0,$$

then  $u_1, \dots, u_n$  can be solved for in terms of  $x_1, \dots, x_n$  and the partial derivatives of  $u_1, \dots, u_n$  can be determined by differentiating implicitly.

# 4 Series Expansions and Transforms

## Taylor Series

$$f(x + \Delta x) = f(x) + \Delta x \frac{df}{dx}(x) + \frac{1}{2} \Delta x^2 \frac{d^2 f}{dx^2}(x) + \cdots + \frac{1}{n!} \Delta x^n \frac{d^n f}{dx^n}(x) + \cdots$$

## Common Series Expansions

$$\begin{aligned} e^x &= \sum_{i=0}^{\infty} \frac{1}{i!} x^i &&= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5) \\ \sin(x) &= \sum_{i=0}^{\infty} (-1)^i \frac{1}{(2i+1)!} x^{2i+1} &&= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + O(x^9) \\ \cos(x) &= \sum_{i=0}^{\infty} (-1)^i \frac{1}{(2i)!} x^{2i} &&= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + O(x^8) \\ \sinh(x) &= \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} x^{2i+1} &&= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + O(x^9) \\ \cosh(x) &= \sum_{i=0}^{\infty} \frac{1}{(2i)!} x^{2i} &&= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + O(x^8) \\ \frac{1}{1-x} &= \sum_{i=0}^{\infty} x^i &&= 1 + x + x^2 + x^3 + x^4 + O(x^5) \\ \ln(1+x) &= \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i &&= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5) \\ \sqrt{1+x} &= \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \left(-\frac{1}{2}\right) \cdots \left(-\frac{1}{2} - i\right) x^i &&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + O(x^5) \end{aligned}$$

## Arithmetic and Geometric Series

$$\sum_{k=0}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=0}^n r^k = \frac{1-r^{n+1}}{1-r}.$$

## Binomial Theorem

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

## Fourier Series

For a periodic function with period  $L$ , we have

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \\ a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \\ b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx. \end{aligned}$$

## Complex Form of the Fourier Series

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{L}\right), \\ c_n &= \frac{1}{2L} \int_{-L}^L f(x) \exp\left(-\frac{in\pi x}{L}\right) dx, \\ &= \begin{cases} \frac{1}{2}(a_n - ib_n), & \text{when } n \neq 0, \\ a_0 & \text{when } n = 0, \\ \frac{1}{2}(a_{-n} + ib_{-n}), & \text{when } n \leq -1. \end{cases} \end{aligned}$$

## Fourier Transform and Inverse Transform

$$\begin{aligned} \mathcal{F}(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{i\omega x} d\omega \end{aligned}$$

## Laplace Transform

$$\mathcal{F}(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t) e^{-st} dt.$$

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# 5 Vector Calculus

## Vector Identities

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \\ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})\end{aligned}$$

## Distributive property

$$\begin{aligned}\nabla \cdot (\mathbf{A} + \mathbf{B}) &= \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \\ \nabla \times (\mathbf{A} + \mathbf{B}) &= \nabla \times \mathbf{A} + \nabla \times \mathbf{B}\end{aligned}$$

## Zero Identities

$$\begin{aligned}\nabla \times (\nabla \psi) &= 0 \\ \nabla \cdot (\nabla \times \mathbf{A}) &= 0\end{aligned}$$

## Product rules for a scalar field

$$\begin{aligned}\nabla(\psi\phi) &= \phi\nabla\psi + \psi\nabla\phi \\ \nabla \cdot (\psi\mathbf{A}) &= \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A} \\ \nabla \times (\psi\mathbf{A}) &= \psi(\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla\psi\end{aligned}$$

## Laplacian of a scalar

$$\nabla^2\psi = \nabla \cdot (\nabla\psi)$$

## Gradient of a square

$$\frac{1}{2}\nabla\mathbf{A}^2 = \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) = \mathbf{A} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{A}.$$

## Curl of a curl

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

## Product rule for a vector dot product

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

## Product rule for a vector cross product

$$\begin{aligned}\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\ \nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}\end{aligned}$$

## Green's theorem

Let  $\mathcal{C}$  be a positively oriented, piecewise smooth, simple closed curve in the plane  $\mathbf{R}^2$ , and let  $\mathcal{D}$  be the region bounded by  $\mathcal{C}$ . If  $L(x, y)$  and  $M(x, y)$  are functions defined on an open region containing  $\mathcal{D}$  and have continuous partial derivatives there, then

$$\oint_{\mathcal{C}} Ldx + Mdy = \iint_{\mathcal{D}} \left( \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dA.$$

## Gauss' divergence theorem

Suppose  $\mathcal{V}$  is a subset of  $\mathbf{R}^n$  (for  $n = 3$ ,  $\mathcal{V}$  represents a volume in 3D space) which is compact and has a piecewise smooth boundary  $\partial\mathcal{V}$ . If  $\mathbf{F}$  is a continuously differentiable vector field on some neighborhood of  $\mathcal{V}$ , then

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{F} dV = \oint_{\partial\mathcal{V}} \mathbf{F} \cdot \hat{\mathbf{n}} dS.$$

## Stokes' curl theorem

Suppose  $\mathcal{S}$  is an oriented, compact surface with boundary  $\partial\mathcal{S}$  in  $\mathbf{R}^3$ . Suppose  $\mathbf{F}$  is a continuously differentiable vector field on some simply connected neighborhood of  $\mathcal{S} \cup \partial\mathcal{S}$ , then

$$\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial\mathcal{S}} \mathbf{F} \cdot d\boldsymbol{\ell}.$$

## Integration by Parts

Suppose  $\Omega$  is an open bounded subset of  $\mathbf{R}^n$  with a piecewise smooth boundary  $\partial\Omega$ . If scalar field  $u$  and vector field  $\mathbf{v}$  are continuously differentiable on  $\Omega \cup \partial\Omega$  then

$$\int_{\Omega} \nabla u \cdot \mathbf{v} dV = \int_{\partial\Omega} u\mathbf{v} \cdot \hat{\mathbf{n}} dS - \int_{\Omega} u\nabla \cdot \mathbf{v} dV.$$

# 6 Thermodynamics

## Gas constant

$$R = 8.314\,472\text{ J kg}^{-1}\text{ K}^{-1}$$

## First Law of Thermodynamics

$$dU = \delta Q - \delta W = \delta Q - pdV.$$

## Second Law of Thermodynamics

$$dQ = TdS.$$

## Ideal Gas Law

$$pV = nRT$$

## Heat capacity (const pressure)

$$\begin{aligned} C_p &= \left(\frac{\partial Q}{\partial T}\right)_p = T \left(\frac{\partial S}{\partial T}\right)_p \\ &= \left(\frac{\partial U}{\partial T}\right)_p + p \left(\frac{\partial V}{\partial T}\right)_p. \end{aligned}$$

## Heat capacity at (const volume)

$$\begin{aligned} C_V &= \left(\frac{\partial Q}{\partial T}\right)_V = T \left(\frac{\partial S}{\partial T}\right)_V \\ &= \left(\frac{\partial U}{\partial T}\right)_V. \end{aligned}$$

## Ratio of specific heats

$$\gamma = \frac{C_p}{C_V}$$

## Enthalpy

$$H \equiv U + pV = \mu N + TS.$$

## Helmholtz Free Energy

$$A \equiv U - TS = \mu N - pV.$$

## Gibbs Free Energy

$$\begin{aligned} G &\equiv U + pV - TS \\ &= H - TS = \mu N. \end{aligned}$$

## Maxwell Relations

$$\left(\frac{\partial T}{\partial V}\right)_{S,N} = - \left(\frac{\partial p}{\partial S}\right)_{V,N}$$

$$\left(\frac{\partial T}{\partial V}\right)_{p,N} = - \left(\frac{\partial p}{\partial S}\right)_{T,N}$$

$$\left(\frac{\partial T}{\partial p}\right)_{S,N} = \left(\frac{\partial V}{\partial S}\right)_{p,N}$$

$$\left(\frac{\partial T}{\partial p}\right)_{V,N} = \left(\frac{\partial V}{\partial S}\right)_{T,N}$$

## Additional Relations

$$\left(\frac{\partial T}{\partial S}\right)_V = \frac{T}{C_V} \quad \left(\frac{\partial T}{\partial S}\right)_p = \frac{T}{C_p}$$

$$\left(\frac{\partial S}{\partial U}\right)_{V,N} = \frac{1}{T} \quad \left(\frac{\partial S}{\partial V}\right)_{N,U} = \frac{p}{T}$$

## Thermodynamic Efficiency (for a heat engine)

$$\eta \equiv \frac{\text{work}}{\text{heat in}}.$$

## Polytropic Gas Law (isentropic process)

$$\frac{p}{\rho^\gamma} = \text{const.}$$

## Isobaric Processes

$$dp = 0, \quad \delta Q = C_p dT = \gamma dU$$

## Isothermal processes

$$dT = 0, \quad \delta Q = -Vdp = pdV = \delta W$$

## Isochoric (incompressible) processes

$$dV = 0, \quad \delta Q = C_V dT = dU$$

## Adiabatic processes

$$\delta Q = 0, \quad C_p dT = Vdp, \quad C_V dT = -pdV$$

# 7 Fluid Mechanics

Material derivative

$$\frac{DX}{Dt} = \frac{\partial X}{\partial t} + \mathbf{u} \cdot \nabla X$$

Steady incompressible Bernoulli equation

$$\frac{1}{2}u^2 + w + gz = \text{constant along streamline}$$

Continuity equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) &= 0 \end{aligned}$$

Time-dependent irrotational Bernoulli equation  
(potential flow)

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + w + gz = 0$$

Euler Fluids (adiabatic)

$$\rho \frac{Du_i}{Dt} = -\nabla_i p, \quad \frac{Ds}{Dt} = 0.$$

Circulation

$$\Gamma = \oint_c \mathbf{v} \cdot d\ell$$

Navier-Stokes equations (incompressible)

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla_j(\rho u_i u_j) = \rho \frac{Du_i}{Dt} = \underbrace{-\nabla_i p}_{\text{pressure}} + \underbrace{\rho g_i}_{\text{gravity}} + \underbrace{\nu \nabla^2 u_i}_{\text{viscosity}} + \underbrace{\frac{\partial \tau_{ij}}{\partial x_j}}_{\text{body forces}}$$

Energy conservation equation

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + \rho \epsilon \right) + \nabla \cdot \left[ \rho \mathbf{u} \left( \frac{1}{2} u^2 + \epsilon + \frac{p}{\rho} \right) \right] = \underbrace{\rho \mathbf{g} \cdot \mathbf{u}}_{\text{gravity}} + \underbrace{\frac{\partial}{\partial x_j} (u_i \tau_{ij})}_{\text{surface forces}} + \underbrace{q}_{\text{heating}}$$

Mechanical energy equation

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 \right) + \nabla \cdot \left( \rho \mathbf{u} \cdot \frac{1}{2} u^2 \right) = \underbrace{-\mathbf{u} \cdot \nabla p}_{\text{pressure}} + \underbrace{\rho \mathbf{g} \cdot \mathbf{u}}_{\text{gravity}} + \underbrace{\frac{\partial}{\partial x_j} (u_i \tau_{ij})}_{\text{surface forces}} - \underbrace{\frac{1}{2} \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2}_{\text{viscous heating}}$$

Thermal energy equation

$$\frac{\partial}{\partial t}(\rho \epsilon) + \nabla \cdot (\mathbf{u} \rho \epsilon) = \underbrace{-p(\nabla \cdot \mathbf{u})}_{\text{expansion/compression}} + \underbrace{q}_{\text{heating}} + \underbrace{\frac{1}{2} \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2}_{\text{viscous heating}}$$

Nondimensional Constants

$Ra \equiv \frac{g\alpha(T - T_\infty)d^3}{\nu\kappa}$	(Rayleigh)	$Re \equiv \frac{UL}{\nu}$	(Reynolds)
$Pr \equiv \frac{\nu}{\kappa}$	(Prandtl)	$Ri \equiv \frac{gh}{U^2}$	(Richardson)
$Fr \equiv \frac{U}{\sqrt{gH}}$	(Froude)	$Ro \equiv \frac{U}{Lf}$	(Rossby)
$Ta \equiv \frac{4\Omega^2 R^2}{\nu^2}$	(Taylor)	$M \equiv \frac{v_0}{v_s}$	(Mach)

# 8 Clouds and Precipitation

Constants for dry air (near 0 °C)

$$\begin{aligned}\rho_a &= 28.96g \text{ mol}^{-1} \\ R_a &= 287J \text{ kg}^{-1} \text{ K}^{-1} \\ c_{pa} &= 1005J \text{ kg}^{-1} \text{ K}^{-1} \\ c_{va} &= 718J \text{ kg}^{-1} \text{ K}^{-1} \\ \kappa &= R_a/c_{pa} = 0.286\end{aligned}$$

Constants for water vapor (near 0 °C)

$$\begin{aligned}\rho_v &= 18.01g \text{ mol}^{-1} \\ R_v &= 461.5J \text{ kg}^{-1} \text{ K}^{-1} \\ c_{pv} &= 1870J \text{ kg}^{-1} \text{ K}^{-1} \\ c_{vv} &= 1410J \text{ kg}^{-1} \text{ K}^{-1} \\ L_f &= 3.340 \times 10^5 J \text{ kg}^{-1} \\ L_v &= 2.260 \times 10^6 J \text{ kg}^{-1}\end{aligned}$$

Ratio of Specific Gas Constants

$$\epsilon = R_a/R_v = 0.622$$

Potential Temperature

$$\theta = T \left( \frac{100kPa}{p} \right)^\kappa$$

Clausius-Clapeyron Equation

$$\frac{de_s}{dT} = \frac{S_2 - S_1}{V_2 - V_1} = \frac{L_v e_s}{R_v T^2}$$

Clausius-Clapeyron Equation (Bolton)

For  $T$  in degrees C and  $e_s$  in mbar,

$$e_s = 6.112 \exp \left[ \frac{17.67 \cdot T}{T + 243.5} \right],$$

Water Vapor Partial Pressure

$$e = \rho_v R_v T = \rho_v \frac{R_a}{\epsilon} T$$

Mixing Ratio

$$w \equiv \frac{m_v}{m_d} = \frac{\rho_v}{\rho_d} = \epsilon \frac{e}{p - e}$$

Specific Humidity

$$q \equiv \frac{m_v}{m_d + m_v} = \frac{\rho_v}{\rho_d + \rho_v} = \epsilon \frac{e}{p - (1 - \epsilon)e}$$

Virtual Temperature

$$T_v = T \left[ \frac{1 + w/e}{1 + w} \right]$$

Equivalent Temperature

$$T_e = T + \frac{L_v}{c_p} q$$

Equivalent Potential Temperature

$$\theta_e \approx T \left( \frac{p_0}{p} \right)^\kappa \exp \left( \frac{L_v q}{c_p T_{LCL}} \right)$$

Pseudoadiabatic Temperature Lapse Rate

$$\frac{dT}{dz} = \frac{\Gamma_d \left( 1 + \frac{L_v q_s}{R_d T} \right)}{\left( 1 + \frac{L_v^2 q_s}{c_p R_v T^2} \right)}$$

Scale Height

$$p = p_0 \exp \left( -\frac{1}{H} (z - z_0) \right)$$

Energy Equation

$$d \left( \frac{1}{2} u^2 + gz \right) + \alpha dp = -\mathbf{f} \cdot d\mathbf{l}$$

Pressure Change for a Convective Vortex

$$\Delta p = p_\infty \left[ 1 - \exp \left( \frac{\gamma \eta (c_p \Delta T + L_v \Delta q)}{(\gamma \eta - 1) RT} \right) \right]$$



# 9 Radiative Transfer

## Planck Constant

$$h = 6.626 \times 10^{-34} \text{ J sec}$$

## Boltzmann Constant

$$k = 1.3806 \times 10^{-23} \text{ J K}^{-1}$$

## Monochromatic Intensity

$$I_\lambda = \frac{dE_\lambda}{\cos \theta d\Omega d\lambda dt dA}$$

## Monochromatic Flux Density

$$F_\lambda = \int_\Omega I_\lambda \cos \theta d\Omega = \int_0^{2\pi} \int_0^{\pi/2} I_\lambda(\theta, \phi) \cos \theta \sin \theta d\theta d\phi$$

## Monochromatic Flux Density (Isotropic)

$$F_\lambda = \pi I_\lambda$$

## Absorptivity

$$A(\nu) = \frac{I_{\nu, \text{abs}}(\nu)}{I_{\nu, \text{incident}}(\nu)}$$

## Emissivity

$$\epsilon(\nu) = \frac{\text{Emission}}{\text{Blackbody Emission}}$$

## Transmissivity

$$\mathcal{T}_\nu = \exp(-\tau(\nu))$$

## Planck's Blackbody Radiation Law

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/k\lambda T} - 1}$$

## Stefan-Boltzmann Law

$$F = \sigma T_e^4$$

## Wien's Displacement Law

$$T/\nu_{max} = \text{const} \quad T\lambda_{max} = \text{const}$$

## Pressure Broadening

$$k_\nu = \frac{S}{\pi} \frac{\alpha}{(\nu - \nu_0)^2 + \alpha^2} = Sf(\nu - \nu_0)$$

$$\alpha = \alpha_0(p/p_0)(T_0/T)^n$$

## Doppler Broadening

$$k_\nu = \frac{S}{\alpha_D \sqrt{\pi}} \exp \left[ - \left( \frac{\nu - \nu_0}{\alpha_D} \right)^2 \right]$$

$$\alpha_D = \nu_0 (2KT/mc^2)^{1/2}$$

## Optical Depth

$$\chi = \int_{s_1}^{s_2} \rho(s) k_m(\nu, s) ds, \quad \tau = \chi \cos \theta$$

## Beer's Law

$$dI_\lambda \propto -I_\lambda ds$$

## General Radiative Transfer Equation

$$\frac{dI_\lambda}{d\tau} = I_\lambda - J_\lambda$$

## Radiative Transfer Equation

$$\frac{dI_\lambda}{d\tau} = I_\lambda(\Omega) - (1-\omega)B_\lambda(T) - \frac{\omega}{4\pi} \int_{4\pi} I_\lambda(\Omega') P_\lambda(\Omega, \Omega') d\Omega'$$

## Asymmetry Factor

$$g = \frac{1}{2} \int_{-1}^1 P(\mu) \mu d\mu$$

# 10 Atmospheric Dynamics

## Constants

$$\begin{aligned}
 g &= 9.80616 \text{ m/s}^2 \\
 R_d &= 287 \text{ J kg}^{-1} \text{ K}^{-1} \\
 c_v &= 717 \text{ J kg}^{-1} \text{ K}^{-1} \\
 c_p &= 1004 \text{ J kg}^{-1} \text{ K}^{-1} \\
 a &= 6.371 \text{ km} \\
 \Omega &= 7.292 \times 10^{-5} \text{ s}^{-1} \\
 \kappa &= R_d/c_p \approx 0.286
 \end{aligned}$$

## Coriolis Parameter

$$f = 2\Omega \sin \theta$$

## Beta-plane Approximation

$$f = f_0 + \beta y, \quad f_0 = 2\Omega \sin \theta_0, \quad \beta = \frac{2\Omega \cos \theta_0}{a}$$

## Hydrostatic Equation

$$dp = -\rho g dz$$

## Dry Adiabatic Lapse Rate

$$\Gamma_d = -\frac{\partial T}{\partial z} = \frac{g}{c_p}$$

## Brunt-Väisälä Frequency

$$N^2 = \frac{g}{\Theta} \frac{\partial \Theta}{\partial z}$$

## Static Stability Parameter

$$S_p = \left( \frac{R_d T}{p c_p} - \frac{\partial T}{\partial p} \right) = -\frac{T}{\Theta} \frac{\partial \Theta}{\partial p} = \frac{\Gamma_d - \Gamma}{\rho g}$$

## Hypsometric Equation

$$z_T = (z_2 - z_1) = \frac{R_d}{g} \int_{p_2}^{p_1} \frac{T}{p} dp$$

## Thermal Wind (Pressure Coordinates)

$$\begin{aligned}
 \mathbf{v}_T = \mathbf{v}_g(p_2) - \mathbf{v}_g(p_1) &= \frac{R_d}{f} \mathbf{k} \times \nabla_p \langle T \rangle \ln \left( \frac{p_1}{p_2} \right) \\
 &= \frac{1}{f} \mathbf{k} \times \nabla_p (\Phi_2 - \Phi_1)
 \end{aligned}$$

## Conservation of PV (Barotropic Fluid)

$$\frac{D_h}{Dt} \left( \frac{\zeta_g + f}{h} \right) = 0$$

## Phase and Group Velocity (1D)

$$v_p = \frac{\nu}{k} \quad v_g = \frac{\partial \nu}{\partial k}$$

## Equations of Motion

$$\begin{aligned}
 \frac{d\rho}{dt} &= -\rho \nabla \cdot \mathbf{v} \\
 \frac{du}{dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi - 2\Omega w \cos \phi \\
 \frac{dv}{dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi \\
 \frac{dw}{dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + 2\Omega u \cos \phi \\
 \frac{c_p}{T} \frac{dT}{dt} - \frac{R_d}{p} \frac{dp}{dt} &= \frac{J}{T} \\
 p &= \rho R_d T
 \end{aligned}$$

## Hydrostatic Equations of Motion

$$\begin{aligned}
 \frac{D \mathbf{v}_h}{Dt} + f \mathbf{k} \times \mathbf{v}_h &= -\nabla_p \Phi \\
 \frac{\partial \Phi}{\partial p} &= -\frac{R_d T}{p} \\
 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} &= 0 \\
 \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - S_p \omega &= \frac{J}{c_p} \\
 p &= \rho R_d T
 \end{aligned}$$

## Quasi-Geostrophic Equations

$$\begin{aligned}
 \frac{D_g \mathbf{v}_g}{Dt} &= -f_0 \mathbf{k} \times \mathbf{v}_a - \beta y \mathbf{k} \times \mathbf{v}_g \\
 \mathbf{v}_g &= \frac{1}{f_0} \mathbf{k} \times \nabla \Phi \quad \left( \zeta_g = \frac{1}{f_0} \nabla^2 \Phi \right) \\
 \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} &= 0 \\
 \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left( -\frac{\partial \Phi}{\partial p} \right) - \sigma \omega &= \frac{\kappa J}{p}
 \end{aligned}$$

## Shallow Water Equations

$$\begin{aligned}
 \frac{du}{dt} - f v + g \frac{\partial}{\partial x} (h + h_t) &= 0 \\
 \frac{dv}{dt} + f u + g \frac{\partial}{\partial y} (h + h_t) &= 0 \\
 \frac{dh}{dt} + h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0
 \end{aligned}$$

## Energy (per unit area)

$$\begin{aligned}
 \text{Internal} \quad I &= \int_0^\infty \rho c_v T dz \\
 \text{Potential} \quad P &= \int_0^\infty \rho g z dz \\
 \text{Total potential} \quad TPE &= I + P \\
 \text{Kinetic} \quad K &= \int_0^\infty \frac{\rho}{2} |\mathbf{v}|^2 dz
 \end{aligned}$$

# 11 Numerical Methods

## Finite-Difference Formulas (Centered)

Difference	Formula	Error term
$f'(x_i)$	$\frac{f_{i+1} - f_{i-1}}{2h}$	$-\frac{1}{6}h^2 f^{(3)}$
	$\frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12h}$	$\frac{1}{30}h^4 f^{(5)}$
	$\frac{f_{i+3} - 9f_{i+2} + 45f_{i+1} - 45f_{i-1} + 9f_{i-2} - f_{i-3}}{60h}$	$-\frac{1}{140}h^6 f^{(7)}$
$f''(x_i)$	$\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$	$-\frac{1}{12}h^2 f^{(4)}$
	$\frac{-f_{i+2} + 16f_{i+1} - 30f_i + 16f_{i-1} - f_{i-2}}{12h^2}$	$\frac{1}{90}h^4 f^{(6)}$
	$\frac{2f_{i+3} - 27f_{i+2} + 270f_{i+1} - 490f_i + 270f_{i-1} - 27f_{i-2} + 2f_{i-3}}{180h^2}$	$-\frac{1}{560}h^6 f^{(8)}$
$f^{(3)}(x_i)$	$\frac{f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}}{2h^3}$	$-\frac{1}{4}h^2 f^{(5)}$
	$\frac{-f_{i+3} + 8f_{i+2} - 13f_{i+1} + 13f_{i-1} - 8f_{i-2} + f_{i-3}}{8h^3}$	$\frac{7}{120}h^4 f^{(7)}$
$f^{(4)}(x_i)$	$\frac{f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}}{h^4}$	$-\frac{1}{6}h^2 f^{(6)}$
	$\frac{-f_{i+3} + 12f_{i+2} - 39f_{i+1} + 56f_i - 39f_{i-1} + 12f_{i-2} - f_{i-3}}{6h^4}$	$\frac{7}{240}h^4 f^{(8)}$

## Finite-Difference Formulas (Uncentered)

Difference	Formula	Error term	Formula	Error term
$f'(x_i)$	$\frac{f_{i+1} - f_i}{h}$	$-\frac{1}{2}hf^{(2)}$	$\frac{f_i - f_{i-1}}{h}$	$\frac{1}{2}hf^{(2)}$
	$\frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h}$	$\frac{1}{3}h^2 f^{(3)}$	$\frac{3f_i - 4f_{i-1} + f_{i-2}}{2h}$	$\frac{1}{3}h^2 f^{(3)}$
	$\frac{2f_{i+3} - 9f_{i+2} + 18f_{i+1} - 11f_i}{6h}$	$-\frac{1}{4}h^3 f^{(4)}$	$\frac{11f_i - 18f_{i-1} + 9f_{i-2} - 2f_{i-3}}{6h}$	$\frac{1}{4}h^3 f^{(4)}$
	$\frac{-f_{i+2} + 6f_{i+1} - 3f_i - 2f_{i-1}}{6h}$	$\frac{1}{12}h^3 f^{(4)}$	$\frac{2f_{i+1} + 3f_i - 6f_{i-1} + f_{i-2}}{6h}$	$-\frac{1}{12}h^3 f^{(4)}$
	$\frac{-3f_{i+4} + 16f_{i+3} - 36f_{i+2} + 48f_{i+1} - 25f_i}{12h}$	$\frac{1}{5}h^4 f^{(5)}$	$\frac{25f_i - 48f_{i-1} + 36f_{i-2} - 16f_{i-3} + 3f_{i-4}}{12h}$	$\frac{1}{5}h^4 f^{(5)}$
	$\frac{f_{i+3} - 6f_{i+2} + 18f_{i+1} - 10f_i - 3f_{i-1}}{12h}$	$-\frac{1}{20}h^4 f^{(5)}$	$\frac{3f_{i+1} + 10f_i - 18f_{i-1} + 6f_{i-2} - f_{i+3}}{12h}$	$-\frac{1}{20}h^4 f^{(5)}$
$f''(x_i)$	$\frac{f_{i+2} - 2f_{i+1} + f_i}{h^2}$	$-hf^{(3)}$	$\frac{f_i - 2f_{i-1} + f_{i-2}}{h^2}$	$hf^{(3)}$
	$\frac{-f_{i+3} + 4f_{i+2} - 5f_{i+1} + 2f_i}{h^2}$	$\frac{11}{12}h^2 f^{(4)}$	$\frac{2f_i - 5f_{i-1} + 4f_{i-2} - f_{i-3}}{h^2}$	$\frac{11}{12}h^2 f^{(4)}$
	$\frac{11f_{i+4} - 56f_{i+3} + 114f_{i+2} - 104f_{i+1} + 35f_i}{12h^2}$	$-\frac{5}{6}h^3 f^{(5)}$	$\frac{35f_i - 104f_{i-1} + 114f_{i-2} - 56f_{i-3} + 11f_{i-4}}{12h^2}$	$\frac{5}{6}h^3 f^{(5)}$
	$\frac{-f_{i+3} + 4f_{i+2} + 6f_{i+1} - 20f_i + 11f_{i-1}}{12h^2}$	$\frac{1}{12}h^3 f^{(5)}$	$\frac{11f_{i+1} - 20f_i + 6f_{i-1} + 4f_{i-2} - f_{i-3}}{12h^2}$	$-\frac{1}{12}h^3 f^{(5)}$
$f^{(3)}(x_i)$	$\frac{f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i}{h^3}$	$-\frac{3}{2}hf^{(4)}$	$\frac{f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}}{h^3}$	$\frac{3}{2}hf^{(4)}$
	$\frac{f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}}{h^3}$	$-\frac{1}{2}hf^{(4)}$	$\frac{f_{i+1} - 3f_i + 3f_{i-1} - f_{i-2}}{h^3}$	$\frac{1}{2}hf^{(4)}$
	$\frac{-3f_{i+4} + 14f_{i+3} - 24f_{i+2} + 18f_{i+1} - 5f_i}{2h^3}$	$\frac{7}{4}h^2 f^{(5)}$	$\frac{5f_i - 18f_{i-1} + 24f_{i-2} - 14f_{i-3} + 3f_{i-4}}{2h^3}$	$\frac{7}{4}h^2 f^{(5)}$
	$\frac{-f_{i+3} + 6f_{i+2} - 12f_{i+1} + 10f_i - 3f_{i-1}}{2h^3}$	$\frac{1}{4}h^2 f^{(5)}$	$\frac{3f_{i+1} - 10f_i + 12f_{i-1} - 6f_{i-2} + f_{i-3}}{2h^3}$	$\frac{1}{4}h^2 f^{(5)}$
$f^{(4)}(x_i)$	$\frac{f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_i}{h^4}$	$-2hf^{(5)}$	$\frac{f_i - 4f_{i-1} + 6f_{i-2} - 4f_{i-3} + f_{i-4}}{h^4}$	$2hf^{(5)}$
	$\frac{f_{i+3} - 4f_{i+2} + 6f_{i+1} - 4f_i + f_{i-1}}{h^4}$	$-hf^{(5)}$	$\frac{f_{i+1} - 4f_i + 6f_{i-1} - 4f_{i-2} + f_{i-3}}{h^4}$	$hf^{(5)}$