MATH 186: Homework 7
Winter 2008

Due: Friday, Mar. 21 before class

> "In mathematics you don't understand things. You just get used to them."
> —John von Neumann

From the textbook:

The problems in parentheses are warm-up problems. You won't turn these in, but they are good practice and can help with the graded problems. The problems in **bold** are the graded problems that you will hand in.

§8.8: (1, 5, 7, 9, 15, 17–24, 26–30, 32, 33, 37) **6, 10, 16, 25, 31**

The problems below are "extra": if you do only the textbook problems in bold above (and do them correctly) you will receive full credit for this homework. You should only work on these after you've completed the textbook problems.

Outside the box:

1. Define the following (very well-known) sequence $\{f_n\}$ by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$. Clearly this sequence does not converge. However, the sequence of its ratios does! Let $r_n = \frac{f_{n+1}}{f_n}$. Write out the first few terms of the sequence. Is this a monotone sequence?

   a) Prove that $\{r_n\}$ converges.

   b) Compute $\lim_{n\to\infty} r_n$.

Practice problems for the exam (Some quite hard. Not to be handed in.) The problems below are meant to supplement the material from sections 7.1, 7.2, 7.5, 7.8, 7.14, 8.1–8.8. The suggested problems from the homework for these sections is another good source of practice problems.

1. Curves $r = f(\theta) = ae^{-b\theta}$ (in polar coordinates) where $a, b > 0$ and $0 \leq \theta < \infty$ are called *logarithmic spirals*. For every $a, b > 0$, calculate the area enclosed by the curve (and x-axis). Calculate also the length of the spiral as an improper integral.

2. Let $r^2 = f(\theta)^2 = \cos^2(\theta) + \tan^2(\theta)$ for $-\pi/2 < \theta < \pi/2$ be a curve in polar coordinates. Calculate the area enclosed by the curve and the y-axis as an improper integral.

3. Calculate the surface area of a solid of revolution, about the x-axis, of the graph of $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = \sin(x)$.

4. Let $s > 0$ be an integer and $f\colon [0, 1] \to \mathbb{R}$ be the function $f(x) = x^s$. Consider the solid of revolution of f about x-axis. Calculate the volume of the solid. Show also that surface area of the solid is at least π for every s.

5. Let $f\colon [a, b] \to [a, b]$ is continuously differentiable, increasing and (strictly) concave. Show that the volume of the solid of revolution about x-axis is larger than the solid of revolution about y-axis, but both solids have the same area.

6. Find a function $f\colon [3, \infty) \to \mathbb{R}$ so that the improper integral $\int_3^\infty f(x)dx$ exists but f has no limit at ∞.

7. Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous function so that the improper integral $\int_5^\infty f(x)dx$ exists. Show that $\int_{2^n}^{2^{n+1}} f(x)dx \to 0$ as $n \to \infty$. Show also that $\int_m^{2m} f(x)dx \to 0$ as $m \to \infty$.

8. Let $f\colon [0,\infty) \to \mathbb{R}$ be a function so that the improper integral $\int_0^\infty f(x)dx$ converges. Let (x_n) be a sequence so that $x_n > 0$ for every $n \geq 0$. Give a proof or a counterexample to the following questions.

   (a) If (x_n) is a monotonic sequence, does the series $\sum_{n=0}^\infty f(x_n)$ converge?

   (b) Does the same series converge if $x_n \to \infty$ as $n \to \infty$?

   (c) Does the series converge if f is monotone and $x_n \geq n$ for every n?

   (d) Does the series converge if $x_n \geq n$ for every n but f is not assumed to be monotonic?

   (e) Does the series converge if f is monotonic and there exists $0 < c < 1$ so that $x_n \geq cn$ for every n?

9. Let $a_n = (-1)^{n+1}/n$ for every $n \geq 1$. Let $S_m = \sum_{n=1}^m a_n$ for every $m \geq 1$. Show that

   (a) Series $\sum_{k=0}^\infty \left(\frac{1}{2k-1} - \frac{1}{2k}\right)$ converges.

   (b) (S_{2m}) converges.

   (c) (S_m) is a Cauchy sequence.

   (d) $\sum_{n=1}^\infty (-1)^{n+1}\frac{1}{n}$ converges.

10. Let

$$a_n = \frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{2n-1}{2n}$$

for every $n \geq 1$. Show that

   (a) $\sum_{k=1}^\infty \ln \frac{2k}{2k-1} = \infty$.

   (b) $\ln a_n \to -\infty$ as $n \to \infty$.

   (c) $a_n \to 0$ as $n \to \infty$.

11. Find a number $c > 0$ and a sequence (λ_n) so that $0 < \lambda_n < 1$ for every n and $\lambda_1\lambda_2\cdots\lambda_n \geq c$ for every n.