Potential energy diagrams

Consider an arbitrary potential energy shown schematically below.

There are a number of important qualitative features of the behavior of the system that can be determined by just knowing this curve. The first thing to notice is that since the kinetic energy of a particle must be greater than or equal to zero, the total energy of the system must be greater than the potential energy. This means that the coordinates of a particle (position, Energy) can never be in the hashed region. This is called the forbidden region of the diagram.

When dealing with an isolated mechanical system the total energy will be constant represented by a horizontal line.
The particle can only be at the regions shown in red. In the graph above there are two allowed regions, the particle must be located somewhere in the red. Once we know where the particle is, then you can determine the force on it and its kinetic energy.

The particle is represented by the red point. It’s position is \( r_0 \) and the kinetic energy is the difference between the total energy (red line) and the potential energy. The force on the particle is the negative of the slope \( F_x = -\frac{dU}{dx} \). So you can determine the speed and the acceleration of the particle, but not the direction of motion. The particle as shown above will oscillate back and forth stopping at the turning points, \( tp_i \) and \( tp_r \). This is a bound trajectory. If the particle were in the right side region it would be in an unbound orbit, visiting the single turning point at most once.

One last thing that you can immediately determine are the points of equilibrium. An equilibrium point is a point where the force on the particle equals zero. In the situation above there are five equilibrium points. Further, there are two types of equilibrium points, stable and unstable. A particle perturbed from a point of stable equilibrium will experience a force acting to return it to the equilibrium point. This leads to oscillatory motion around the stable equilibrium point.

The converse of this is if the particle is disturbed and experiences a force that continues to accelerate its motion away from the point of equilibrium. Then this is an unstable equilibrium. Stable equilibria occur at local minima of the potential energy, while unstable equilibria occur at local maxima.

It is clear that a lot of information about the possible motions can be gleaned by simply examining the potential energy diagram.

To continue the discussion, consider the potential energy for two particles interacting gravitationally, \( U_G(r) = -\frac{Gm_1m_2}{r} \). The graph of this is shown below.
The first thing to note is that the potential energy, as defined, is always negative, but since only changes in potential energy are physically meaningful this is fine. Consider the particles changing their separation from 10 to 2. Then we can read off the graph \( \Delta U = U(2) - U(10) \approx (-5) - (-1) = -4 \), so the potential energy decreased. If on the other hand, the particle separation went from 2 to 10 then \( \Delta U = U(10) - U(2) \approx (-1) - (-5) = 4 \) and the potential energy of the particles has increased. So both increases and decreases in potential energy of the system can be realized.

Now consider the special case of circular motion of a light particle around a stationary massive one. Let the radius of the orbit be \( r \). What are the total energy, potential energy and kinetic energy for this system? Since the motion will be much less than the speed of light we can ignore the rest energy of the particles, then the total energy will be \( E = KE + U = KE + \left( -G \frac{m_1 m_2}{r} \right) \).

For a circular orbit we can find the kinetic energy through \( F_G = m_1 \frac{v^2}{r} \Rightarrow \frac{1}{2} m_1 v^2 = \frac{1}{2} r F_G \).

Therefore the kinetic energy equals \( KE = \frac{1}{2} m_1 v^2 = \frac{1}{2} r G \frac{m_1 m_2}{r^2} = \frac{1}{2} G \frac{m_1 m_2}{r} \). Putting this together for the total energy yields:

\[
E = \frac{1}{2} G \frac{m_1 m_2}{r} - G \frac{m_1 m_2}{r} = -\frac{1}{2} G \frac{m_1 m_2}{r}. \quad \text{The total energy is less than zero!}
\]

How does this appear on the potential energy diagram?
On this diagram the orbit of \( m_1 \) about \( m_2 \) is represented as a point with the coordinates 
\[(r_0, E) = (r_0, -G \frac{m_1 m_2}{2r})\]. From the graph we can tell that the orbit has a fixed radius \( r_0 \) and a 
constant kinetic and potential energies. The potential energy is 
\[U(r_0) = -G \frac{m_1 m_2}{r}\] and the kinetic 
ergy is the difference between the total energy and the potential energy,

\[KE = E - U(r_0) = G \frac{m_1 m_2}{2r}\]. Note that twice the kinetic energy is equal to the negative of the 
potential energy, \(2KE = -U\), this is a specific case of a very general theorem, the virial theorem in mechanics.

In the above special case of circular motion we were able to solve for all the energies easily. 
This is not often the case, and it is in these circumstances that a potential energy diagram is most 
helpful. Consider an elliptical orbit, for this case the diagram looks like this:
Again the total energy $E$ is a constant, as the system is isolated. However this time the particle separation varies from the perihelion ($r_p$, closest approach) to the aphelion ($r_a$, farthest point). Along with this comes a variation in both the kinetic energy and the potential energy such that their sum is always constant.

What does a rocket leaving the surface of a planet look like on such a diagram?
Before blast off the kinetic energy of the rocket is zero. To escape the gravitational attraction of the planet the rocket must move from the surface to infinity. The amount of energy needed to move the rocket to infinity and arrive with zero velocity is called the escape energy. This amount of energy is shown on the graph as the brace, it is defined as \( KE_{\text{escape}} = -U(r_{\text{surface}}) \). If you want to escape the planet and have a nonzero velocity, say to go to a farther galaxy, then the rocket must be able to supply more energy than this.

For an object of mass \( m \) to escape from the earth the kinetic energy needed is

\[
KE_{\text{escape Earth}} = G \frac{mM_{\text{Earth}}}{r_{\text{earth}}}.
\]

More typically people look at the escape velocity defined as

\[
\frac{1}{2}mv_{\text{escape}}^2 = KE_{\text{escape Earth}} = G \frac{mM_{\text{Earth}}}{r_{\text{earth}}} \quad \text{and so} \quad v_{\text{escape}} = \sqrt{\frac{2GM_{\text{Earth}}}{r_{\text{earth}}}}.
\]

Note that this velocity does not depend on the mass of the object.

There are a couple general things that can be seen from the potential energy diagrams. If the total energy of the system is less than \( U(\infty) = 0 \) then the mass cannot escape and this is called a bound system. If the total energy of the system is zero the mass can reach infinity with zero velocity. Lastly, if the total energy is greater than zero the system is unbound and eventually the two particles will separate.

Moving to unbound orbits, there are two general ways to have them. The first we have already considered, that is the KE is too large to be bound. This is represented below.
The particle is above to travel to infinity. (There is a detail here that is being over looked and that is conservation of angular momentum. In three dimensions the singularity at zero particle separation is removed by including the effects of angular momentum.)

The other way to have an unbound orbit is to have a repulsive potential as will occur between two charges of like sign. Such a situation is shown below.

This particle is either moving toward or away from a repulsive potential. The closest that it can get is \( r_p \), at this point the particle has zero kinetic energy and reverses its motion and accelerates away from the turning point, never to return.

Before leaving gravitational potential energy we need to examine the case when the size of the mass cannot be neglected. For distances less than the radius of one of the masses we know that the force is linear in distance from the center. This gives a potential energy that is parabolic with distance. The expression for the potential energy inside of a mass with radius \( R \) is

\[
U_g(r < R) = \frac{1}{2} \frac{GmM}{R^3} r^2 - \frac{3}{2} \frac{GmM}{R}.
\]

This function smoothly matches with the potential energy outside of the mass. In the graph below the radius of the mass is 2 and the red dot shows the value of the potential energy at that position.
The potential energy function can be approximated near the surface by a linear relationship. This line is shown in red in the next plot. The equation for this line is,

\[ U_{G\text{ linear}}(r \sim R) = -2 \frac{GmM}{R} + \frac{GmM}{R^2} r. \]

This can be rewritten as

\[ U_{G\text{ linear}}(r \sim R) = \frac{GmM}{R^2} (-2R + r) = \frac{GmM}{R^2} (r - R) - \frac{GmM}{R}. \]

Now call the height above the large mass \( h = r - R \) and this becomes

\[ U_{G\text{ linear}}(h) = m \left( \frac{GM}{R^2} \right) h - \frac{GmM}{R}. \]

Remember, the local acceleration due to gravity is \( g = \frac{GM}{R^2} \), then this becomes

\[ U_{G\text{ linear}}(h) = mgh + C. \]

That should look familiar. The constant \( C \) was \(- \frac{GmM}{R}\) in the equation above, but since only differences in potential energy are physically meaningful, this constant can be adjusted set the zero of the potential.

A graphical construction of this is shown below. The red line is a plot of the linear approximation. The slope is \( \Delta U \Delta r = \frac{mg\Delta h}{\Delta h} = mg = \frac{GmM}{R^2} \). Of course, the range of validity is much smaller than shown on the graph.
Turning now to a different situation, consider the potential energy of a harmonic oscillator where the equilibrium point is $x_0 = 3$. 
The total energy of the system is 35 and the two turning points are -1.1 and 7.1. The particle will oscillate between these two extremes with the energy being transferred back and forth between kinetic and potential. The energies are schematically shown below.

The kinetic energy is red, the potential energy is black and the total energy is shown in blue. The kinetic and potential energies oscillate at twice the frequency of motion and so make two cycles while the particle makes one oscillation in the potential well.

Let’s look at a general potential shown below:
The equilibrium point is $r_0 = 3$ and the total energy of the system is guaranteed to be greater than -5. With the total energy as shown (-2.9) the turning points are at 2.5 and 4.1. Although the motion of this particle is oscillatory, this potential is clearly not symmetric or harmonic and the oscillations would not be sinusoidal at this amplitude. With a lower energy the oscillations would more closely resemble those of a harmonic oscillator. This will be a topic for a future discussion.