Center of Momentum Frame

Newton’s laws of motion hold in any inertial frame of reference. However, it is often simplest to examine an isolated system of particles from the frame of reference where the total momentum is zero. Since the system is isolated, the total momentum is constant and in this “Center of Momentum” frame remains zero as the particles interact with one another.

Consider the simplest isolated system containing a single particle. This particle will have a constant momentum in any inertial frame, however if you select the frame that moves along with the particle, the particle will have momentum equal to zero. Can’t get any simpler than that.

Things are considerably more interesting when the isolated system contains two interacting particles. There are two different situations, a bound or an unbound system. Let’s first consider an unbound system. Here the particles are going to eventually end up well separated and no longer interact with one another. If they also start far apart then you can consider this as a collision. A collision starts with the particles isolated and having constant individual momenta, then the particles interact and momentum is transferred, finally the particles separate and have constant individual momenta. A common situation is where one of the particles is initially stationary in the “laboratory” frame of reference. The incoming particle collides with a “fixed target”. Such a collision typically has a trajectory as shown below:

This is a collision between a proton and a He nucleous. The He nucleous is initially at rest and the proton has an initial velocity of 0.05c. As the proton approaches the target, its trajectory is deflected. At the same time the nucleous begins to accelerate away. The black line shows the trajectory of the center of mass (CM). The total momentum of the isolated system is constant and therefore the CM moves to the right with a constant velocity throughout the entire time. The velocity of the CM is given by

$$\vec{V}_{CM} = \frac{\vec{p}_{total}}{m_{total}}.$$

It must be emphasized that the CM moves “before, during and after” the collision. The diagram of the momentum before and after the collision is shown below.
Now we want to change our frame of reference. Imagine moving along with the CM at $V_{CM}$. In this frame the He nucleus moves towards the CM from the right, while the proton moves to the left. Of course, the CM (small black dot in figure) remains stationary in this frame, the so called “Center of Momentum” frame of reference. From this frame of reference the same collision as above is shown on the right.

The diagrams of the momenta before and after the collision are shown above left. The total momentum in this frame of reference is zero and the collision can be viewed as a rotation of the individual momenta. This is an elastic collision so the magnitudes of the momentum vectors are unchanged. If energy had been lost in the collision the momenta would be equal and opposite, but reduced in magnitude from the initial value.

One approach to solving this “scattering” problem is to start in the Lab frame, transform to the Center of Momentum (CoM) frame, solve the interaction for changes in momenta of the two particles and then transform back to the Lab frame to produce the desired result. This procedure is done because it is so much simpler to solve the scattering in the CoM frame, it is a rotation of the two momenta, with a possible change in magnitude if the collision is inelastic. So the question then becomes how does one transform to the CoM frame and back?

For an isolated two particle system the total momentum $\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2$ is a constant. This is true in any frame of reference. To transform between the Lab frame and the CoM from you need to
know the velocity of the CM. This can be found by taking the derivative with respect to time of the CM position, \( \vec{R}_{CM} = \frac{m_1 \vec{r}_{1,Lab} + m_2 \vec{r}_{2,Lab}}{m_1 + m_2} \). Differentiating yields

\[
\vec{v}_{CM} = \frac{m_1 \vec{v}_{1,Lab} + m_2 \vec{v}_{2,Lab}}{m_1 + m_2} = \frac{\vec{p}_{Lab}}{m_1 + m_2}.
\]

To find the velocities of particles one and two in the CoM frame, we subtract the CM velocity from the particles Lab velocity, \( \vec{v}_{1,CM} = \vec{v}_{1,Lab} - \vec{v}_{CM} \) and similarly for particle two. Substituting in the expression for \( \vec{v}_{CM} \) results in

\[
\vec{v}_{1,CM} = \frac{m_2}{m_1 + m_2} (\vec{v}_{1,Lab} - \vec{v}_{2,Lab}) \quad \text{and} \quad \vec{v}_{2,CM} = \frac{m_1}{m_1 + m_2} (\vec{v}_{2,Lab} - \vec{v}_{1,Lab}).
\]

From these we can see that the momenta of the particles in the CoM frame are

\[
\vec{p}_{1,CM} = \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_{1,Lab} - \vec{v}_{2,Lab}) \quad \text{and} \quad \vec{p}_{2,CM} = \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_{2,Lab} - \vec{v}_{1,Lab}).
\]

Clearly these are equal and opposite vectors and so the total momentum in the CoM frame is indeed zero.

If you need to move from the CoM frame to the Lab reference, then you just invert these transformations, \( \vec{v}_{1,Lab} = \vec{v}_{1,CM} + \vec{V}_{CM} \) and \( \vec{v}_{2,Lab} = \vec{v}_{2,CM} + \vec{V}_{CM} \).

**Reduction of two-body problem to an effective one-body problem.**

In the previous discussion we were dealing with two particles acting through a central force, a force that depends on the particle separation and acts along the line between them. Newton’s equations for this system are:

\[
\frac{d\vec{p}_1}{dt} = \vec{F}(r) \quad \text{and} \quad \frac{d\vec{p}_2}{dt} = -\vec{F}(r).
\]

Using the nonrelativistic expression of momentum, we have

\[
m_1 \frac{d^2 \vec{r}_1}{dt^2} = \vec{F}(r) \quad \text{and} \quad m_2 \frac{d^2 \vec{r}_2}{dt^2} = -\vec{F}(r).
\]

Multiplying each equation by the other mass and subtracting them yields:

\[
m_1 m_2 \frac{d^2 (\vec{r}_1 - \vec{r}_2)}{dt^2} = (m_1 + m_2)\vec{F}(r).
\]

Simplifying provides a very interesting and important equation:

\[
\frac{m_1 m_2}{m_1 + m_2} \frac{d^2 (\vec{r}_1 - \vec{r}_2)}{dt^2} = \vec{F}(r).
\]

Defining \( \vec{r} \equiv \vec{r}_1 - \vec{r}_2 \) and \( \mu = \frac{m_1 m_2}{m_1 + m_2} \) and noting that \( r = |\vec{r}| \) yields

\[
\mu \frac{d^2 \vec{r}}{dt^2} = \vec{F}(r).
\]

This is the equation for a single particle with a mass \( \mu \) acted on by the same
force as in the original two particle problem. The two particle problem has been reduced to a
one particle problem with an effective mass, called the “reduced mass” at position \( \vec{r} = \vec{r}_1 - \vec{r}_2 \). So
what has happened to the “other particle”? There is still a second particle in the problem,
however its motion does not depend on the force \( \vec{F}(r) \). How can that be? The second particle is
located at the CM and acts with a mass given by \( M = m_1 + m_2 \), the total mass. This particle can
indeed be in motion, however for an isolated system it moves with a constant velocity \( \vec{V}_{CM} \) as
described previously, independent of \( \vec{F} \).

This simplification is very important, we have taken a two-body problem with both
particles being acted upon by a force and transformed it into two separate one-body
problems with only one of these being acted on by a force. This is an enormous reduction
in complexity. We can solve the one-particle problem of the reduced mass particle,
combine that with the constant velocity of the CM and produce the solution for the two
interacting particles.

The mathematical procedure follows this line. For an isolated system the total momentum
\( \vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 \) is a constant. The other quantity of interest is the relative position, \( \vec{r} = \vec{r}_1 - \vec{r}_2 \),
between the two particles. Taking the derivative of the relative position gives \( \vec{v} = \vec{v}_1 - \vec{v}_2 \). We
can now solve for \( \vec{v}_1 \) and \( \vec{v}_2 \) in terms of the total momentum \( \vec{P} \) and the relative velocity \( \vec{v} \). This
system of two equations and two unknowns yields,

\[
\vec{v}_1 = \frac{\vec{P}}{m_1 + m_2} + \frac{m_2}{m_1 + m_2} \vec{v} \quad \text{and} \quad \vec{v}_2 = -\frac{\vec{P}}{m_1 + m_2} + \frac{m_1}{m_1 + m_2} \vec{v}.
\]

It is convenient to express this in
terms of the reduced mass \( \mu = \frac{m_1 m_2}{m_1 + m_2} \) and the velocity of the CM, \( \vec{V}_{CM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} = \frac{\vec{P}}{m_1 + m_2} \).

Rewriting the expressions for the individual particle velocities in terms of the reduced mass and
velocity of the CM gives:

\[
\vec{v}_1 = \vec{V}_{CM} + \frac{\mu}{m_1} \vec{v} \quad \text{and} \quad \vec{v}_2 = \vec{V}_{CM} - \frac{\mu}{m_2} \vec{v} \quad \text{or in terms of momenta} \quad \vec{p}_1 = m_1\vec{V}_{CM} + \mu\vec{v} \quad \text{and} \quad \vec{p}_2 = m_2\vec{V}_{CM} - \mu\vec{v}.
\]

The right hand side of these equations use the effective single particle
problem and CM, while the left hand side is the velocity of the two real particles. Therefore
these equations describe how to go from the motion of the effective one particle problem to that
of the original two particles.
The dynamics of the effective single particle problem are governed by \( \mu \frac{d\vec{v}}{dt} = \vec{F}(r) \). From this you can find \( \vec{v}(t) \), and combined with the motion of the CM, the velocities of the real particles can be found.

A few trajectories will help illustrate the concepts. Consider the binary star system in your homework. You are in a nonaccelerating rocket (lab frame of reference \( \vec{v} \neq \vec{V}_{CM} \)) observing the motion of the stars. You would see something like this.

The stars trace out quite complicated orbits. Now (having taken Physics 160) you know that things are simpler when considered from the CoM frame and so you change course and match your velocity with that of white point on the diagram. This is the CM and it is tracing out a straight path with constant velocity. Remember that this is just a point in space, there is actually no mass located there.

From your new reference frame the orbits of the two stars look quite different. First, they appear to be ellipses with periodic orbits. It was difficult before to identify an
orbital period because the stars did not return to the same point. However, now the system is clearly periodic. Note that the CM is a stationary spot as it should be in the CoM frame of reference.

If we look at the trajectory of $\vec{r} = \vec{r}_1 - \vec{r}_2$, shown left in white, we see that it also traces out an ellipse. This is not an accident, but a reflection of the fact that this problem can be reduced to an effective single particle problem and that this reduced mass particle will trace out an orbit equal to that of $\vec{r} = \vec{r}_1 - \vec{r}_2$.

Consider the single particle problem $\mu \frac{d\vec{v}}{dt} = \vec{F}(r)$. The interaction force is between the reduced mass particle and the CM. The force is given by

$$\vec{F}(r) = -G \frac{\mu M}{r^2} \vec{r} = -G \left( \frac{m_1 m_2}{m_1 + m_2} \right) \frac{m_1 + m_2}{r^2} \vec{r} = -G \frac{m_1 m_2}{r^2} \vec{r},$$

the same force of interaction as between the real particles with masses $m_1$ and $m_2$. The initial position and momentum of the reduced-mass particle can be determined from the initial positions and momenta of the original particles. Specifically,

$$\vec{r}_{\text{init,}\mu} = \vec{r}_{\text{init,}1} - \vec{r}_{\text{init,}2} \quad \text{and} \quad \vec{p}_{\text{init,}\mu} = \mu \vec{v}_{\text{init}} = \frac{m_2 \vec{p}_{\text{init,}1} - m_1 \vec{p}_{\text{init,}2}}{m_1 + m_2}.$$
previous figure. In this figure the white dot is the location of the fixed CM and is at the one of the foci of the ellipse. This is not the case in the previous figure. In that figure the CM is stationary, but does not rest at a focus of either of the red orbital ellipses.

The simplification of the two-body problem to a single particle with an effective mass and the constant motion of the center-of-mass is an important concept in mechanics. We will see this effective mass reduction return very soon for oscillating systems.