The Lorentz factor \( \gamma \)

Einstein’s theory of relativity supersedes Newtonian mechanics. However, that does not mean that all mechanics problems need to be solved relativistically. Newton’s second law still holds:

\[
\frac{d\vec{P}}{dt} = \sum \vec{F},
\]

however the definition of the momentum for a particle is changed to

\[
\vec{P} = \frac{m\vec{v}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}},
\]

where \( c \) is the speed of light in a vacuum. This new factor, the Lorentz factor,

\[
\gamma(v) = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.
\]

occurs so frequently in the theory of special relativity that it is given its own symbol, gamma \( \gamma(v) \). The first thing to do when you come across a new function is to examine its behavior. For speeds low in comparison to the speed of light, \( c \approx 3 \times 10^8 \text{ m/s} \), we can neglect the \( v/c \) term and then \( \lim_{c \to 0} \gamma = 1 \). This just returns us to the Newtonian definition for momentum, \( \vec{P} = m\vec{v} \). At the other extreme, as the speed of the particle approaches \( c \), \( \gamma \) diverges to infinity and clearly Newtonian mechanics is not valid. So at what intermediate speeds does one need to begin worrying about relativistic effects? Of course, the answer depends on how accurately you need to know the answer. But first let’s graph \( \gamma(v/c) \).

So for small velocities, gamma is indeed close to 1.
To do better than this we can use a Taylor series expansion to more closely approximate gamma. Recall that a function can be expanded around any point by an infinite series containing derivatives of the function evaluated at the point that is being expanded around. Mathematically this is expressed as:

\[
f(x_0 + \delta x) = f(x_0) + f'(x_0)\delta x + \frac{1}{2} f''(x_0)\delta x^2 + \frac{1}{6} f'''(x_0)\delta x^3 + \ldots + \frac{1}{n!} f^n(x_0)\delta x^n + \ldots
\]

where the behavior of \( f \) in the neighbor of \( x \) is to be approximated and \( \delta \) defines the range of the approximation.

Let’s take a look at this polynomial approximation term by term. In the figure below the red line is a sketch of the function to be approximated, the point of interest, \( x_0 + \delta x \), is shown in red and the point that is being expanded about, \( x_0 \), is black. The zeroth order approximation to the function is \( f(x_0 + \delta x) \approx f(x_0) \). This just says that near the point of interest the function can be approximated as a constant, shown in blue. The error made by this approximation is the vertical distance between the correct value, red dot and the approximation blue dot. This might be a good enough approximation for some situations, but we do better.

The next term in the series is:

\[
f'(x_0)\delta x = \left. \frac{df}{dx} \right|_{x_0} \delta x = \delta f .
\]

This is the slope of the function multiplied by the distance away for the selected point. With the addition of this term in the approximation, now \( f(x_0 + \delta x) = f(x_0) + f'(x_0)\delta x \), this approximates the function as a line with non-zero slope, shown in green. The value for this approximation is the green dot and the error is the difference between the red dot and the green. The error is reduced and clearly the green line is a better approximation to the function than is the blue one.
Let’s examine a couple common functions. First find an approximation for \( \exp(x) \) where \( x \) is near 3. The expansion for the function is:

\[
\exp(x)|_{x=3} \approx \exp(3) + \exp(3)(x-3) + \frac{1}{2}\exp(3)(x-3)^2 + \frac{1}{6}\exp(3)(x-3)^3
\]

Below are graphs of the zeroth (orange), first (green), second (red) and third (blue) order approximation around \( x = 3 \). The true function is shown in black, but is so well fit by the third order curve that you cannot see it.

The error decreases as the order of the approximation is increased, same colors as above.
A similar expansion is shown below for \(\sin(x)\) around \(x = 1\). The second order approximation is shown in red. Typically in physics the second order expansion is used. The reason for this will become apparent later in the semester.

![Graph of \(\sin(x)\) with second order approximation shown in red.]

Returning to the problem at hand, the understanding of 
\[
\gamma(v) \equiv \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}},
\]
in this case we are concerned about low speeds, or \(x_0 = 0\), and \(\delta x = \left(\frac{v}{c}\right)^2\), therefore we need to find an approximation for 
\[
\frac{1}{\sqrt{1 - \delta x}}.
\]

Using the expression above
\[
f(\delta x) = f(0) + f'(0)\delta x + \frac{1}{2} f''(0)\delta x^2 + \frac{1}{6} f'''(0)\delta x^3 + \ldots + \frac{1}{n!} f^n(0)\delta x^n
\]
Let’s look at the expression term by term. The first term says that \(f(\delta x) \approx f(0)\) and since \(f(0) = 1\) this is just the simplest approximation \(\gamma \approx 1\). This Newtonian approximation will often be good enough, but how much better can we do by adding the next term in the expansion?

Look at \(f(\delta x) = 1 + f'(0)\delta x\), to evaluate this we need the derivative 
\[
\frac{df(\delta x)}{d\delta x} = \frac{d}{d\delta x} \left(\frac{1}{\sqrt{1 - \delta x}}\right).
\]
(Note that \(\delta x = \left(\frac{v}{c}\right)^2\) so the radical is of \((1 - \delta x)\) not \((1 - \delta x^2)\)). Performing the derivative gives
\[ \frac{df(\delta x)}{d\delta x} = \frac{d}{d\delta x} \left( \frac{1}{\sqrt{1-\delta x}} \right) = \frac{1}{2} \left( \frac{1}{(1-\delta x)^{3/2}} \right) \] and we need this evaluated at \( \delta x = 0 \) and so

\[ \left. \frac{df(\delta x)}{d\delta x} \right|_{\delta x=0} = \frac{1}{2}. \]

Finally \( f(\delta x) = 1 + \frac{1}{2} \delta x \) or \( \gamma(v) \approx 1 + \frac{1}{2} \left( \frac{v}{c} \right)^2 \) this is the next level of approximation for gamma. I leave it to you to show that the third level of approximation is

\[ \gamma(v) \approx 1 + \frac{1}{2} \left( \frac{v}{c} \right)^2 + \frac{3}{8} \left( \frac{v}{c} \right)^4. \]

The general series can be found from

\[ \gamma(v) = \sum_{n=0}^{\infty} \left[ \prod_{k=1}^{n} \left( \frac{2k-1}{2k} \right) \right] \left( \frac{v}{c} \right)^{2n} \]

where the first factor in the brace gives the coefficient and the second yields the even power terms of \( \left( \frac{v}{c} \right) \).

Let’s plot the Newtonian gamma (red), the first (orange) and second (green) approximations and gamma (black) on the same graph.

You can see that the first and second approximations bend upward and match the true function for higher speeds better than \( \gamma \approx 1 \). If we plot the difference between the approximations and gamma we can more closely examine the error. Using the same color scheme the graph is below.
The red curve is the difference between the Newtonian approximation and gamma. The horizontal line is at 1% error. You can see that up to 4% of the speed of light the error is very small. Four percent of the speed of light is \( \sim 1 \times 10^7 \text{ m/s} \), extremely fast for macroscopic objects. The graph below better shows the error for such low speeds. The \( \gamma \approx 1 \) approximation is in red. The error is small, but look how much better the Taylor series expansion does.

This is why for most everyday objects Newtonian mechanics is sufficient. However, as the speed increases toward \( c \) the error starts to grow rapidly. The first order approximation (orange) reduces this error considerably out until \( \sim 0.2c \) and the second order terms extend the approximations out to \( \sim 0.4c \).
Let’s look at one last point. Perhaps the most famous equation in physics is Einstein’s $E = mc^2$. This expression points out the equivalence between mass and energy. When a body is at rest it has a mass and therefore an energy, these are termed rest mass and rest energy. If the body is put into motion, work is done and the total energy increases. This used to be thought of as the mass of the body increasing. This relativistic mass depended on the velocity. Einstein predicted that the relativistic mass would vary with velocity as, $m_{\text{rel}} = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma m_0$, where $m_0$ is the rest mass of the particle. So as the velocity of a particle increased so would its relativistic mass and it would increase by the factor $\gamma$. Therefore the total energy of the particle could be written as $E = \gamma m_0 c^2$. If we think of the total energy being the sum of the rest energy, $E_0 = m_0 c^2$ and the energy of motion, the kinetic energy, we have $E = KE + E_0$ or $\gamma m_0 c^2 = KE + m_0 c^2$. Solving for the kinetic energy yields $KE = (\gamma - 1)m_0 c^2$.

What does this reduce to when we use the low velocity expression for $\gamma$? Clearly, $\gamma \approx 1$ is not useful, as it just says that the $KE \approx 0$. To do better we can use $\gamma(v) \approx 1 + \frac{1}{2} \left(\frac{v}{c}\right)^2$, this means the kinetic energy is $KE = \left(1 + \frac{1}{2} \left(\frac{v}{c}\right)^2\right) m_0 c^2 = \frac{1}{2} m_0 v^2$. This is just the familiar classical expression for kinetic energy. It is very reassuring to be able to show that the Newtonian kinetic energy can be found from the low velocity limit of Einstein’s relativistic expression for the total energy of a particle.

The use of the velocity varying mass $m_{\text{ref}}$ has fallen out of favor with physicists for a number of reasons. Nowadays the mass of a particle is taken always to be the rest mass so the subscript on $m_0$ is dropped and $m$ is used.
# Plots gamma, several approximations to gamma and their absolute errors

from __future__ import division

from visual import *
from visual.graph import *  # import graphing features

gamma_function = gdisplay(  title = 'gamma vs. speed/c',
    xtitle = 'v/c',
    ytitle = 'gamma',
    foreground=color.black,
    background= color.white)  # set display #1

error = gdisplay(  x=0, y=400,   # plot the next display down from the top
    title = 'error vs. speed/c',
    xtitle = 'v/c', ymax =.05,
    ytitle = 'gamma - approx_gamma (%)',
    foreground=color.black,
    background= color.white)  # set display #2

gamma_func = gcurve(gdisplay = gamma_function, color = color.black)  # curve for gamma
one = gcurve(gdisplay = gamma_function, color = color.red)    # Newtonian approx gamma = 1
approx_func_2 = gcurve(gdisplay = gamma_function, color = color.orange) # curve for gamma = 1 + (v/c)**2
approx_func_4 = gcurve(gdisplay = gamma_function, color = color.green)  # curve for gamma = 1 + (v/c)**2 + 0.375(v/c)**3

approx_func_0 = gcurve(gdisplay = error, color = color.red)         # curve for Newtonian i.e. gamma =
delta_gamma = gcurve(gdisplay = error, color = color.orange)      # gamma - (1+(v/c)**2) in percent
delta_gamma_2 = gcurve(gdisplay = error, color = color.green)       # gamma - (1+(v/c)**2 + 0.375(v/c)**3) in percent
one_percent = gcurve(gdisplay = error, color = color.black)       # draw line at 1% for error

t = 0.  # scaled speed in terms of c i.e. v/c
dv = .005  # graphing increment

while (v < 1):
    # for v in arange (0, 0.999, dv):
    # Could use for loop instead of while

        gamma = 1/sqrt(1-v**2)
        approx_gamma = 1 + 0.5*(v**2)
        approx_gamma_4 = 1 + 0.5*(v**2) + 3/8*(v**4)

        gamma_func.plot (pos=(v, gamma ))        # plot real gamma
        one.plot            (pos=(v, 1 ))            # plot gamma = 1
        approx_func_2.plot  (pos=(v, approx_gamma))   # plot gamma = 1 + .5(v/c)**2
        approx_func_4.plot  (pos=(v, approx_gamma_4))       # plot gamma = 1 + .5(v/c)**2 + .375(v/c)**3

    if v > .2:   # limit the range of the second plot to .2c
        pass # do nothing, not interested in this range for second graph
    else:
        delta_gamma.plot (pos = (v, 100*(gamma - approx_gamma)/gamma)) # difference in percent
        approx_func_0.plot (pos = (v, 100*(gamma - 1)/gamma)) # difference between classical and special relativity
        one_percent.plot    (pos = (v, 1))  # 1% error line
        delta_gamma_2.plot (pos = (v, 100*(gamma - approx_gamma_4)/gamma))

v = v + dv  # advance to next point on graphs