

Hopf Bifurcation Calculations in Delayed Systems with Translational Symmetry

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Summary. The Hopf bifurcation of an equilibrium in dynamical systems consisting of n equations with a single time delay and translational symmetry is investigated. The Jacobian belonging to the equilibrium of the corresponding delay-differential equations always has a zero eigenvalue due to the translational symmetry. This eigenvalue does not depend on the system parameters, while other characteristic roots may satisfy the conditions of Hopf bifurcation. An algorithm for this Hopf bifurcation calculation (including the center-manifold reduction) is presented. The closed-form results are demonstrated for a simple model of cars following each other along a ring.

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1. Introduction

The generalization of the bifurcation theory of ordinary differential equations (ODEs) to delay-differential equations (DDEs) is summarized in the book of Hale and Verduyn Lunel [8]. The corresponding normal form theorem is published by Hale et al. in [7] showing some examples, too. A theoretical review of Hopf bifurcation in DDE systems is also available in the book of Diekmann et al. [5]. In the case of the simplest scalar first-order nonlinear DDE, the first closed-form Hopf bifurcation calculation was carried out by Hassard et al. in [9], while for a vector DDE, Stépán presented such calculations

in [17], showed other applications in [18], and presented a closed-form codimension-two Hopf bifurcation calculation in [19].

Because of the complexity of calculations, many researchers tried to compile computer algebra programs for detecting and analyzing Hopf bifurcations in DDEs. For example, Campbell and Bélair constructed a Maple program [3]. As a result of the Hopf bifurcation algorithm, the first Fourier approximation of stable or unstable periodic orbits can be derived analytically as a function of the bifurcation parameters. This estimate is very useful in many applications, especially when the limit cycle is unstable. However, it is acceptable only for bifurcation parameters close enough to the critical point, since Taylor series expansion of the nonlinearity up to third order is used in the DDE.

Engelborghs et al. solve the same task numerically with a Matlab package DDE-BIFTOOL [6]. This program can follow branches of stable and unstable orbits against the chosen bifurcation parameters. Since it is a seminumerical method using the exact form of the nonlinearities, it provides reliable results even far away from the critical bifurcation parameter. Moreover, it can also be used when a system includes more than one delay.

The goal of this paper is an analytical bifurcation analysis of Hopf bifurcations in DDEs. The presence of translational symmetry in the nonlinear equations gives rise to a relevant zero eigenvalue in the linearized system at any of the trivial solutions. It happens in a way similar to that in the case of the so-called compartment systems presented by Krisztin in [12]. This property causes singularities in the standard Hopf bifurcation calculations when two further characteristic roots cross the imaginary axis. The corresponding linear algebraic equations occurring in the Hopf bifurcation algorithm cannot be solved due to the steady zero characteristic root. This causes major difficulties when the algorithm is implemented in symbolic manipulation (such as Maple or Mathematica). To avoid this problem, we give the Hopf bifurcation calculation for these systems after subtracting the subspace related to the translational symmetry.

The method is demonstrated for a simple car-following model with translational symmetry along a ring and a constant time delay, namely the reflex time of the drivers. The presence of a robust subcritical Hopf bifurcation is shown in the example, which gives a hint why traffic jams often develop into stop-and-go motion. We note that this kind of symmetry can also be found in the dynamics of semiconductor lasers near a *continuous wave* state, as shown by Verduyn Lunel and Krauskopf [20].

2. Retarded Functional Differential Equations with Translational Symmetry

Dynamical systems that are described by so-called retarded or delay-differential equations have memory: The rate of change of the present states depend on the past states of the system. Time development of these systems can be described by retarded functional differential equations (RFDEs). When translational symmetry occurs in a delayed dynamical system, any of its motion can be shifted by constant values, in the following sense.

Let us consider the special nonlinear RFDE in the form

$$\dot{x}(t) = f(Gx_t; \mu), \quad (1)$$

where the state variable is $x: \mathbb{R} \rightarrow \mathbb{R}^n$, the dot refers to the derivative with respect to the time t , and the function $x_t: \mathbb{R} \rightarrow \mathbb{X}_{\mathbb{R}^n}$ is defined by the shift $x_t(\vartheta) = x(t + \vartheta)$, $\vartheta \in [-r, 0]$, where the length of the delay $r \in \mathbb{R}^+$ is assumed to be finite. The linear functional $G: \mathbb{X}_{\mathbb{R}^n} \rightarrow \mathbb{R}^n$ acts on the function space $\mathbb{X}_{\mathbb{R}^n}$ of $\mathbb{R} \rightarrow \mathbb{R}^n$ functions. For the sake of simplicity, let the bifurcation parameter be the scalar $\mu \in \mathbb{R}$, and then let the function $f: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ be analytic, and

$$f(0; \mu) = 0, \quad (2)$$

for any μ . Thus the trivial solution $x(t) \equiv 0$ of the RFDE (1) exists for all the values of the bifurcation parameter. Since the space $\mathbb{X}_{\mathbb{R}^n}$ is infinite-dimensional, the dimension of the phase space of RFDE (1) also becomes infinite.

According to the *Riesz Representation Theorem*, the linear functional G has the general form defined by the *Stieltjes integral*

$$Gx_t = \int_{-r}^0 d\gamma(\vartheta)x(t + \vartheta), \quad (3)$$

where the $n \times n$ matrix $\gamma: [-r, 0] \rightarrow \mathbb{R}^{n \times n}$ is a function of bounded variation.

The translational symmetry of the system (1) is expressed by the following property of the linear functional G given in (3):

$$\text{Ker} \left(\int_{-r}^0 d\gamma(\vartheta) \right) \neq \{0\} \Leftrightarrow \det \int_{-r}^0 d\gamma(\vartheta) = 0. \quad (4)$$

Consequently, if there is a solution $\tilde{x}(t)$ of (1) for a certain parameter μ , then $\tilde{x}(t) + c$ is also a solution if the constant vector $c \in \mathbb{R}^n$ satisfies $Gc = 0$ or, equivalently, the linear homogeneous algebraic equation

$$\int_{-r}^0 d\gamma(\vartheta)c = 0. \quad (5)$$

Indeed,

$$\frac{d}{dt}(\tilde{x}(t) + c) = \dot{\tilde{x}}(t), \quad (6)$$

and

$$f(G(\tilde{x}_t + c); \mu) = f(G\tilde{x}_t + Gc; \mu) = f(G\tilde{x}_t; \mu), \quad (7)$$

which is implied by (4). Condition (4) also implies that infinitely many vectors c satisfy (5).

In other words, $x(t) \equiv 0$ is not the only trivial solution of RFDE (1). Any solution $x(t) \equiv c$ satisfies (1) for all the parameter values μ since

$$f(Gc; \mu) = f \left(\int_{-r}^0 d\gamma(\vartheta)c; \mu \right) = f(0; \mu) = 0 \quad (8)$$

is satisfied by infinitely many vectors c due to the property (4).

The above-described class of delayed systems can be generalized further for systems governed by

$$\dot{x}(t) = f_1(G_1x_t; \mu) + f_2(G_2x_t; \mu). \quad (9)$$

These systems also have translational symmetry if the two linear functionals satisfy

$$\text{Ker} \left(\int_{-r}^0 d\gamma_1(\vartheta) \right) \cap \text{Ker} \left(\int_{-r}^0 d\gamma_2(\vartheta) \right) \neq \{0\}, \quad (10)$$

which implies that the corresponding determinants are zero:

$$\det \int_{-r}^0 d\gamma_1(\vartheta) = 0, \quad \det \int_{-r}^0 d\gamma_2(\vartheta) = 0. \quad (11)$$

However, it is the condition (10) that guarantees that infinitely many constant vectors c satisfy $G_1c = 0$ and $G_2c = 0$. Consequently, if there is a solution $\tilde{x}(t)$ of (9) for a certain parameter μ , then $\tilde{x}(t) + c$ is also a solution.

3. Stability and Bifurcations

The linearization of RFDE (1) at any of its trivial solutions c in (5) results in the variational system

$$\dot{x}(t) = \int_{-r}^0 d_\vartheta \eta(\vartheta; \mu)x(t + \vartheta), \quad (12)$$

where the $n \times n$ matrix function $\eta: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is defined by

$$\eta(\vartheta; \mu) = D_x f(0; \mu)\gamma(\vartheta), \quad (13)$$

and the $n \times n$ matrix $D_x f$ is the derivative of f . Clearly, condition (4) yields

$$\det \int_{-r}^0 d_\vartheta \eta(\vartheta; \mu) = \det \left(D_x f(0; \mu) \int_{-r}^0 d\gamma(\vartheta) \right) = 0, \quad (14)$$

for all values of the bifurcation parameter μ .

Similar to the case of linear ODEs, the substitution of the trial solution $x(t) = ke^{\lambda t}$ into (12) with a constant vector $k \in \mathbb{C}^n$ and characteristic exponent $\lambda \in \mathbb{C}$ results in the characteristic equation

$$D(\lambda; \mu) = \det \left(\lambda I - \int_{-r}^0 d_\vartheta \eta(\vartheta; \mu)e^{\lambda\vartheta} \right) = 0. \quad (15)$$

Among the infinitely many characteristic exponents, there is

$$\lambda_0(\mu) \equiv 0, \quad (16)$$

for any μ since η satisfies (14). If the multiplicity of the zero characteristic exponent is only 1, the corresponding eigenvector spans the linear one-dimensional eigenspace

embedded in the infinite-dimensional phase space of the nonlinear RFDE (1). Along this the trivial solutions $x(t) \equiv c$ satisfying condition (5) are located. In the same way, possible corresponding high-dimensional subspaces can also be identified for the more general case (9).

Obviously, these trivial solutions of the nonlinear RFDE (1) cannot be asymptotically stable for any bifurcation parameter μ . Still, they can be stable in the Lyapunov sense if all the other infinitely many characteristic exponents are situated in the left half of the complex plane. Also, Hopf bifurcations may occur in the complementary part of the phase space with respect to the eigenspace of the zero exponent if there exist pure imaginary characteristic exponents at some critical parameter values μ_{cr} :

$$\lambda_{1,2}(\mu_{cr}) = \pm i\omega. \quad (17)$$

In the parameter space of the RFDE, the corresponding stability boundaries are described by the so-called D -curves

$$R(\omega) = \operatorname{Re} D(i\omega; \mu_{cr}) = 0, \quad S(\omega) = \operatorname{Im} D(i\omega; \mu_{cr}) = 0, \quad (18)$$

that are parameterized by the frequency $\omega \in \mathbb{R}^+$ referring to the imaginary part of the above critical characteristic exponents (17). Since (15) has infinitely many solutions for λ , an ∞ -dimensional version of the *Routh-Hurwitz* criterion is needed to decide on which side of the D -curves the steady state is stable or unstable. These investigations will be based on [18], but other criteria are also available in the literature [11], [14], [16].

Another condition of the existence of Hopf bifurcation is the nonzero speed of the critical characteristic exponents $\lambda_{1,2}$ in (17) when they cross the imaginary axis due to the variation of the bifurcation parameter μ :

$$\operatorname{Re} \left(\frac{d\lambda_{1,2}(\mu_{cr})}{d\mu} \right) = \operatorname{Re} \left(-\frac{\partial D(\lambda; \mu_{cr})}{\partial \mu} \left(\frac{\partial D(\lambda; \mu_{cr})}{\partial \lambda} \right)^{-1} \right) \neq 0. \quad (19)$$

This can be checked by implicit differentiation of the characteristic function (15).

The super- or subcritical nature of the Hopf bifurcation, that is, the stability and estimated amplitudes of the periodic motions arising about the stable or unstable trivial solutions can be determined via the investigation of the third-degree power series of the original nonlinear RFDE (1). The above conditions (17) and (19) can be checked using the variational system (12) independently from the zero characteristic exponent (16). In contrast, the lengthy calculation with the nonlinear part leads to unsolvable singular equations if the eigenspace corresponding to the zero exponent is not removed.

In the subsequent sections, the type of the Hopf bifurcation is determined when a zero characteristic exponent exists due to the translational symmetry in the nonlinear system (1) induced by (4), or equivalently by (10). The algorithm will be presented when a single discrete time delay τ occurs in the delayed dynamical system.

4. Hopf Bifurcation in Case of One Discrete Delay and Translational Symmetry

The following analysis is based on [17], [18]. However, the calculations are carried out for an arbitrary number of DDEs and also for the case of a singular Jacobian caused by a translational symmetry as explained above.

Consider the following autonomous nonlinear system with one discrete delay $\tau \in \mathbb{R}^+$:

$$\dot{x}(t) = \Lambda x(t) + Px(t - \tau) + \Phi(x(t), x(t - \tau)), \quad (20)$$

where, according to (4), the constant matrices $\Lambda, P \in \mathbb{R}^{n \times n}$ satisfy

$$\det(\Lambda + P) = 0. \quad (21)$$

The near-zero analytic function $\Phi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is supposed to keep the translational symmetry, that is,

$$\Phi(x(t) + c, x(t - \tau) + c) = \Phi(x(t), x(t - \tau)), \quad \text{for all } c \neq 0: (\Lambda + P)c = 0. \quad (22)$$

Note that condition (22) is fulfilled, for example, by

$$\Phi(x(t), x(t - \tau)) = \tilde{\Phi}(\Lambda x(t) + Px(t - \tau)), \quad (23)$$

when system (20) is considered in the form of (1) satisfying conditions (4), and consequently (5).

Introduce the dimensionless time $\tilde{t} = t/\tau$. Characteristic exponents and associated frequencies are also transformed as $\tilde{\lambda} = \tau\lambda$ and $\tilde{\omega} = \tau\omega$, respectively. By abuse of notation, we drop the tildes immediately in the transformed form of equation (20):

$$\dot{x}(t) = \tau\Lambda x(t) + \tau Px(t - 1) + \tau\Phi(x(t), x(t - 1)). \quad (24)$$

Hereafter, consider the time delay τ as the bifurcation parameter μ . This is a natural choice in applications where the mathematical models are extended by modelling delay effects. Of course, the calculations below can still be carried out in the same way if different bifurcation parameters are chosen.

The characteristic equation of (24) assumes the form

$$D(\lambda; \tau) = \det(\lambda I - \tau\Lambda - \tau P e^{-\lambda}) = 0. \quad (25)$$

Condition (21) implies that the zero exponent (16) exists, that is,

$$\lambda_0(\tau) \equiv 0 \quad (26)$$

is always a characteristic root.

Suppose that the necessary conditions (17) and (19) are also fulfilled, i.e., there exists a critical time delay τ_{cr} such that

$$\lambda_{1,2}(\tau_{\text{cr}}) = \pm i\omega, \quad \text{Re} \left(\frac{d\lambda_{1,2}(\tau_{\text{cr}})}{d\tau} \right) \neq 0, \quad (27)$$

while all the other characteristic exponents $\lambda_k, k = 3, 4, \dots$ are situated in the left half of the complex plane when the time delay is in a finite neighborhood of its critical value.

4.1. Operator Differential Equation

The dimensionless delay-differential equation (24) can be rewritten in the form of an operator-differential equation (OpDE). For the parameter case of $\tau = \tau_{\text{cr}}$, we obtain

$$\dot{x}_t = \mathcal{A}x_t + \mathcal{F}(x_t), \quad (28)$$

where the dot still refers to differentiation with respect to the time t , and the linear and nonlinear operators $\mathcal{A}, \mathcal{F}: \mathbb{X}_{\mathbb{R}^n} \rightarrow \mathbb{X}_{\mathbb{R}^n}$ are defined as

$$\mathcal{A}\phi(\vartheta) = \begin{cases} \phi'(\vartheta), & \text{if } -1 \leq \vartheta < 0, \\ \mathbf{L}\phi(0) + \mathbf{R}\phi(-1), & \text{if } \vartheta = 0, \end{cases} \quad (29)$$

$$\mathcal{F}(\phi)(\vartheta) = \begin{cases} 0, & \text{if } -1 \leq \vartheta < 0, \\ F(\phi(0), \phi(-1)), & \text{if } \vartheta = 0, \end{cases} \quad (30)$$

respectively. Here, prime stands for differentiation with respect to ϑ , while the $n \times n$ matrices \mathbf{L}, \mathbf{R} , and the nonlinear function F are given as

$$\mathbf{L} = \tau_{\text{cr}}\Lambda, \quad \mathbf{R} = \tau_{\text{cr}}\mathbf{P}, \quad \text{and} \quad F = \tau_{\text{cr}}\Phi. \quad (31)$$

Note that consideration of the first rows of the operators \mathcal{A}, \mathcal{F} on domains of $\mathbb{X}_{\mathbb{R}^n}$ that are restricted by their second rows, gives the same mathematical description (see [5] for details or [20] for discussions).

The translational symmetry is inherited by the OpDE (28), since (21) implies

$$\det(\mathbf{L} + \mathbf{R}) = 0, \quad (32)$$

and similarly, (22) implies that the near-zero nonlinear operator \mathcal{F} satisfies

$$\mathcal{F}(x_t + c) = \mathcal{F}(x_t) \Leftrightarrow F(x(t) + c, x(t-1) + c) = F(x(t), x(t-1)), \quad (33)$$

for all $c \neq 0: (\mathbf{L} + \mathbf{R})c = 0$.

In accordance with (23), condition (33) is fulfilled, for example, by

$$F(x(t), x(t-1)) = \tilde{F}(\mathbf{L}x(t) + \mathbf{R}x(t-1)). \quad (34)$$

Clearly, the operator \mathcal{A} has the same characteristic roots as the linear part of the delay-differential equation (24) at τ_{cr} :

$$\text{Ker}(\lambda\mathcal{I} - \mathcal{A}) \neq \{0\} \Leftrightarrow D(\lambda; \tau_{\text{cr}}) = \det(\lambda\mathbf{I} - \mathbf{L} - \mathbf{R}e^{-\lambda}) = 0, \quad (35)$$

and the corresponding three critical characteristic exponents (26) and (27) are also the same:

$$\lambda_0(\tau) \equiv 0, \quad \lambda_{1,2}(\tau_{\text{cr}}) = \pm i\omega. \quad (36)$$

If the zero root appeared only for the critical bifurcation (actually, the time delay) parameter τ_{cr} , then it would mean that a fold bifurcation occurs together with a Hopf bifurcation, as investigated by Sieber and Krauskopf [15] in the case of a controlled inverted pendulum. In contrast, we consider the case where the determinants (21,32)

hold, and the corresponding zero characteristic exponent (26,36) exists for arbitrary bifurcation parameter τ . In this case, it is impossible to carry out the Hopf bifurcation calculation by disregarding this zero root. More exactly, the center-manifold reduction related to the pure imaginary characteristic roots cannot be carried out by the usual algorithm: A linear nonhomogeneous equation occurs with coefficient matrix $L + R$ that leads to a contradiction (see Section 4.3).

We can avoid the above problem in the phase space if we restrict the system to the complementary (infinite-dimensional) space of the linear one-dimensional invariant manifold spanned by that eigenvector of the operator \mathcal{A} which belongs to the zero eigenvalue. After the construction of the reduced OpDE, the usual Hopf bifurcation calculation algorithm can be carried out, including the center-manifold reduction related to the pure imaginary eigenvalues.

Although the reduction of the OpDE (28) can be carried out for any value of the bifurcation parameter, the calculations are presented for only the critical value, since the subsequent Hopf bifurcation calculations use the system parameters only at the critical values.

4.2. Reduced OpDE

The eigenvector $s_0 \in \mathbb{X}_{\mathbb{R}^n}$ (actually, $s_0: [-1, 0] \rightarrow \mathbb{R}^n$) satisfies

$$\mathcal{A}s_0 = \lambda_0 s_0 \quad \Rightarrow \quad \mathcal{A}s_0 = 0. \quad (37)$$

The definition (29) of the linear operator \mathcal{A} in (37) leads to the simple boundary value problem

$$s_0'(\vartheta) = 0, \quad Ls_0(0) + Rs_0(-1) = 0. \quad (38)$$

Its constant solution is

$$s_0(\vartheta) \equiv S_0 \in \mathbb{R}^n, \quad (L + R)S_0 = 0. \quad (39)$$

In order to project the system to s_0 and to its complementary space, we also need the adjoint operator (see [8]):

$$\mathcal{A}^*\psi(\sigma) = \begin{cases} -\psi'(\sigma), & \text{if } 0 < \sigma \leq 1, \\ L^*\psi(0) + R^*\psi(1), & \text{if } \sigma = 0, \end{cases} \quad (40)$$

where $*$ denotes either adjoint operator or transposed conjugate vector and matrix. The eigenvector $n_0 \in \mathbb{X}_{\mathbb{R}^n}$ of \mathcal{A}^* associated with the $\lambda_0^* = 0$ eigenvalue satisfies

$$\mathcal{A}^*n_0 = \lambda_0^* n_0 \quad \Rightarrow \quad \mathcal{A}^*n_0 = 0. \quad (41)$$

Its solution gives

$$n_0(\sigma) \equiv N_0 \in \mathbb{R}^n, \quad (L^* + R^*)N_0 = 0. \quad (42)$$

Thus, the vectors S_0 and N_0 are the right and left eigenvectors of the matrix $L + R$, respectively, belonging to the zero eigenvalue. The inner product definition

$$\langle \psi, \phi \rangle = \psi^*(0)\phi(0) + \int_{-1}^0 \psi^*(\xi + 1)R\phi(\xi) d\xi \quad (43)$$

is used to calculate the normality condition

$$\langle n_0, s_0 \rangle = 1 \quad \Rightarrow \quad N_0^*(I + R)S_0 = 1, \quad (44)$$

from which one of the two freely eligible scalar values in S_0 , N_0 is determined.

Separate the phase space with the help of the new state variables $z_0: \mathbb{R} \rightarrow \mathbb{R}$ and $x_t^-: \mathbb{R} \rightarrow \mathbb{X}_{\mathbb{R}^n}$ defined as

$$\begin{cases} z_0 = \langle n_0, x_t \rangle, \\ x_t^- = x_t - z_0 s_0. \end{cases} \quad (45)$$

Now the OpDE (28) can be semidecoupled by using the above definitions, the normalized eigenvectors (39,42) satisfying (37,41), the inner product definition (43), and the translational symmetry expressed, for example, by (22,33):

$$\begin{aligned} \dot{z}_0 &= \langle n_0, \dot{x}_t \rangle = \langle n_0, \mathcal{A}x_t + \mathcal{F}(x_t) \rangle \\ &= \langle \mathcal{A}^*n_0, x_t \rangle + \langle n_0, \mathcal{F}(x_t^- + z_0 s_0) \rangle \\ &= n_0^*(0)\mathcal{F}(x_t^- + z_0 s_0)(0) = N_0^*\mathcal{F}(x_t^-)(0), \\ \dot{x}_t^- &= \dot{x}_t - \dot{z}_0 s_0 = \mathcal{A}x_t + \mathcal{F}(x_t) - n_0^*(0)\mathcal{F}(x_t^- + z_0 s_0)(0)s_0 \\ &= \mathcal{A}x_t^- + z_0 \mathcal{A}s_0 + \mathcal{F}(x_t^- + z_0 s_0) - n_0^*(0)\mathcal{F}(x_t^- + z_0 s_0)(0)s_0 \\ &= \mathcal{A}x_t^- + \mathcal{F}(x_t^-) - N_0^*\mathcal{F}(x_t^-)(0)S_0. \end{aligned} \quad (46)$$

In the first part, the scalar differential equation of (46) becomes fully separated, if the equation is restricted to the corresponding manifold spanned by the eigenvector s_0 . $x_t^- = 0$ implies $\dot{z}_0 = 0$; hence, all the trivial solutions $x(t) \equiv c = z_0 s_0$ are situated along a straight line (the corresponding invariant manifold) at any constant z_0 .

In the second part, the operator differential equation of (46) is already fully decoupled, and can be redefined as

$$\dot{x}_t^- = \mathcal{A}x_t^- + \mathcal{F}^-(x_t^-), \quad (47)$$

where the new nonlinear operator \mathcal{F}^- assumes the form

$$\mathcal{F}^-(\phi)(\vartheta) = \begin{cases} -N_0^*\mathcal{F}(\phi)(0)S_0, & \text{if } -1 \leq \vartheta < 0, \\ \mathcal{F}(\phi)(0) - N_0^*\mathcal{F}(\phi)(0)S_0, & \text{if } \vartheta = 0, \end{cases} \quad (48)$$

and after the substitution of definition (30) of the near-zero nonlinear operator \mathcal{F} :

$$\mathcal{F}^-(\phi)(\vartheta) = \begin{cases} -N_0^*F(\phi(0), \phi(-1))S_0, & \text{if } -1 \leq \vartheta < 0, \\ F(\phi(0), \phi(-1)) - N_0^*F(\phi(0), \phi(-1))S_0, & \text{if } \vartheta = 0. \end{cases} \quad (49)$$

While the linear operator remains the same, the reduction of the system related to the translational symmetry does cause change in the nonlinear operator. This change will have an essential role in the center-manifold reduction of the Hopf analysis of OpDE (28) below.

4.3. Center-Manifold Reduction of the Reduced OpDE

The algorithm of the usual Hopf bifurcation analysis is well known and presented in several books [9], [18]. Here, we apply this for the reduced OpDE (47), and only those

steps will be detailed where the new form of the nonlinear operator makes differences relative to the standard case (28) without the zero eigenvalue.

First, let us determine the real eigenvectors $s_{1,2} \in \mathbb{X}_{\mathbb{R}^n}$ of the linear operator \mathcal{A} associated with the critical eigenvalue $\lambda_1 = i\omega$. These eigenvectors satisfy

$$\mathcal{A}s_1 = -\omega s_2, \quad \mathcal{A}s_2 = \omega s_1. \quad (50)$$

After the substitution of definition (29) of \mathcal{A} , these equations form a $2n$ -dimensional coupled linear first-order boundary value problem (similar to (38)):

$$\begin{bmatrix} s_1'(\vartheta) \\ s_2'(\vartheta) \end{bmatrix} = \omega \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} s_1(\vartheta) \\ s_2(\vartheta) \end{bmatrix}, \quad \begin{bmatrix} L & \omega I \\ -\omega I & L \end{bmatrix} \begin{bmatrix} s_1(0) \\ s_2(0) \end{bmatrix} + \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} s_1(-1) \\ s_2(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (51)$$

Its solution is

$$\begin{bmatrix} s_1(\vartheta) \\ s_2(\vartheta) \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \cos(\omega\vartheta) + \begin{bmatrix} -S_2 \\ S_1 \end{bmatrix} \sin(\omega\vartheta), \quad (52)$$

with constant vectors $S_{1,2} \in \mathbb{R}^n$ having two freely eligible scalar variables while satisfying the homogeneous equations

$$\begin{bmatrix} L + R \cos \omega & \omega I + R \sin \omega \\ -(\omega I + R \sin \omega) & L + R \cos \omega \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (53)$$

The eigenvectors $n_{1,2}$ of \mathcal{A}^* associated with $\lambda_1^* = -i\omega$ are determined by

$$\mathcal{A}^*n_1 = \omega n_2, \quad \mathcal{A}^*n_2 = -\omega n_1. \quad (54)$$

The use of definition (40) of \mathcal{A}^* leads to another boundary value problem, which has the solution

$$\begin{bmatrix} n_1(\sigma) \\ n_2(\sigma) \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cos(\omega\sigma) + \begin{bmatrix} -N_2 \\ N_1 \end{bmatrix} \sin(\omega\sigma), \quad (55)$$

where the constant vectors $N_{1,2} \in \mathbb{R}^n$ also possess two freely eligible scalar variables while satisfying

$$\begin{bmatrix} L^* + R^* \cos \omega & -(\omega I + R^* \sin \omega) \\ \omega I + R^* \sin \omega & L^* + R^* \cos \omega \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (56)$$

The orthonormality conditions

$$\langle n_1, s_1 \rangle = 1, \quad \langle n_1, s_2 \rangle = 0 \quad (57)$$

determine two of the four freely eligible scalar values in $S_{1,2}, N_{1,2}$. The application of the inner product definition (43) results in two linear equations, which are arranged for the two free parameters in N_1 and N_2 in the following way:

$$\begin{aligned} & \frac{1}{2} \begin{bmatrix} S_1^* (2I + R^* (\cos \omega + \frac{\sin \omega}{\omega})) + S_2^* R^* \sin \omega & -S_1^* R^* \sin \omega + S_2^* R^* (\cos \omega - \frac{\sin \omega}{\omega}) \\ -S_1^* R^* \sin \omega + S_2^* (2I + R^* (\cos \omega + \frac{\sin \omega}{\omega})) & -S_1^* R^* (\cos \omega - \frac{\sin \omega}{\omega}) - S_2^* R^* \sin \omega \end{bmatrix} \\ & \times \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \quad (58)$$

Note that taking 1 and 0 as first components of the vectors S_1 and S_2 , respectively, are reasonable choices for the two remaining scalar parameters; see Section 5 and also [3].

With the help of the right and left eigenvectors $s_{1,2}$ and $n_{1,2}$ of operator \mathcal{A} , introduce the new state variables

$$\begin{cases} z_1 = \langle n_1, x_t^- \rangle, \\ z_2 = \langle n_2, x_t^- \rangle, \\ w = x_t^- - z_1 s_1 - z_2 s_2, \end{cases} \quad (59)$$

where $z_{1,2}: \mathbb{R} \rightarrow \mathbb{R}$ and $w: \mathbb{R} \rightarrow \mathbb{X}_{\mathbb{R}^n}$. Using the above definitions, the eigenvectors (52,55) satisfying (50,54), the inner product definition (43), and the definition of operator \mathcal{F}^- (48), the reduced OpDE (47) can be rewritten in the form

$$\begin{aligned} \dot{z}_1 &= \langle n_1, \dot{x}_t^- \rangle = \langle n_1, \mathcal{A}x_t^- + \mathcal{F}^-(x_t^-) \rangle = \langle \mathcal{A}^* n_1, x_t^- \rangle + \langle n_1, \mathcal{F}^-(x_t^-) \rangle \\ &= \omega \langle n_2, x_t^- \rangle + n_1^*(0) \mathcal{F}^-(x_t^-)(0) + \int_{-1}^0 n_1^*(\xi + 1) \mathbf{R} \mathcal{F}^-(x_t^-)(\xi) d\xi \\ &= \omega z_2 + n_1^*(0) \mathcal{F}(x_t^-)(0) - \left(n_1^*(0) \mathbf{I} + \int_{-1}^0 n_1^*(\xi + 1) d\xi \mathbf{R} \right) (N_0^* \mathcal{F}(x_t^-)(0) S_0) \\ &= \omega z_2 + \left(N_1^* - \left((N_1^* (\mathbf{I} + \frac{\sin \omega}{\omega} \mathbf{R}) - N_2^* \frac{1 - \cos \omega}{\omega} \mathbf{R}) S_0 \right) N_0^* \right) \mathcal{F}(x_t^-)(0), \\ \dot{z}_2 &= -\omega z_1 + \left(N_2^* - \left((N_1^* \frac{1 - \cos \omega}{\omega} \mathbf{R} + N_2^* (\mathbf{I} + \frac{\sin \omega}{\omega} \mathbf{R})) S_0 \right) N_0^* \right) \mathcal{F}(x_t^-)(0), \\ \dot{w} &= \dot{x}_t^- - \dot{z}_1 s_1 - \dot{z}_2 s_2 = \mathcal{A}x_t^- + \mathcal{F}^-(x_t^-) - \omega z_2 s_1 + \omega z_1 s_2 \\ &\quad - \left(N_1^* - \left((N_1^* (\mathbf{I} + \frac{\sin \omega}{\omega} \mathbf{R}) - N_2^* \frac{1 - \cos \omega}{\omega} \mathbf{R}) S_0 \right) N_0^* \right) \mathcal{F}(x_t^-)(0) s_1 \\ &\quad - \left(N_2^* - \left((N_1^* \frac{1 - \cos \omega}{\omega} \mathbf{R} + N_2^* (\mathbf{I} + \frac{\sin \omega}{\omega} \mathbf{R})) S_0 \right) N_0^* \right) \mathcal{F}(x_t^-)(0) s_2. \end{aligned} \quad (60)$$

The introduction of the new scalar parameters

$$\begin{aligned} q_1 &= \left(N_1^* (\mathbf{I} + \frac{\sin \omega}{\omega} \mathbf{R}) - N_2^* \frac{1 - \cos \omega}{\omega} \mathbf{R} \right) S_0, \\ q_2 &= \left(N_1^* \frac{1 - \cos \omega}{\omega} \mathbf{R} + N_2^* (\mathbf{I} + \frac{\sin \omega}{\omega} \mathbf{R}) \right) S_0 \end{aligned} \quad (61)$$

is related to the translational symmetry in the system, that is, $q_{1,2}$ would be zero if there were no zero characteristic root in the system (28), because in that case $S_0 = 0$. But even if the translational symmetry is there, it is often possible to find $N_1^* \mathbf{R} S_0 = N_2^* \mathbf{R} S_0 = N_1^* S_0 = N_2^* S_0 = 0$ resulting in $q_1 = q_2 = 0$, for example, when $\mathbf{R} S_0 = 0$ also holds apart from $(\mathbf{L} + \mathbf{R}) S_0 = 0$ in (39) (see Section 5).

The structure of the new form of the reduced OpDE (47) is as follows:

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{w} \end{bmatrix} &= \begin{bmatrix} 0 & \omega & \mathcal{O} \\ -\omega & 0 & \mathcal{O} \\ 0 & 0 & \mathcal{A} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ w \end{bmatrix} \\ &\quad + \begin{bmatrix} (N_1^* - q_1 N_0^*) \mathcal{F}(z_1 s_1 + z_2 s_2 + w)(0) \\ (N_2^* - q_2 N_0^*) \mathcal{F}(z_1 s_1 + z_2 s_2 + w)(0) \\ -\sum_{j=1,2} (N_j^* - q_j N_0^*) \mathcal{F}(z_1 s_1 + z_2 s_2 + w)(0) s_j + \mathcal{F}^-(z_1 s_1 + z_2 s_2 + w) \end{bmatrix}, \end{aligned} \quad (62)$$

where $\mathcal{F}(z_1 s_1 + z_2 s_2 + w)(0) = F(z_1 s_1(0) + z_2 s_2(0) + w(0), z_1 s_1(-1) + z_2 s_2(-1) + w(-1))$ according to (30), and this expression also appears in $\mathcal{F}^-(z_1 s_1 + z_2 s_2 + w)$ as defined by (48,49).

We need to expand the nonlinearities in power series form, and to keep only those which result in terms up to third order only after the reduction to the center-manifold. In order to do this, we calculate only the terms having second and third order in $z_{1,2}$ and the terms $z_{1,2} w_i$, ($i = 1, \dots, n$) for $\dot{z}_{1,2}$, while only the second-order terms in $z_{1,2}$ are needed for \dot{w} (see (64), (65)). This calculation is possible directly via the Taylor expansion of the analytic function $F: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ of (31) in the definition of (30) and (49) of the near-zero operators \mathcal{F} and \mathcal{F}^- , respectively. The resulting truncated system of OpDE assumes the form

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{w} \end{bmatrix} &= \begin{bmatrix} 0 & \omega & \mathcal{O} \\ -\omega & 0 & \mathcal{O} \\ 0 & 0 & \mathcal{A} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ w \end{bmatrix} + \begin{bmatrix} \sum_{j,k \geq 0}^{j+k=2,3} f_{jk0}^{(1)} z_1^j z_2^k \\ \sum_{j,k \geq 0}^{j+k=2,3} f_{jk0}^{(2)} z_1^j z_2^k \\ \frac{1}{2} \sum_{j,k \geq 0}^{j+k=2} (f_{jk0}^{(3c)} \cos(\omega\vartheta) + f_{jk0}^{(3s)} \sin(\omega\vartheta)) z_1^j z_2^k \end{bmatrix} \\ &+ \begin{bmatrix} \sum_{i=1}^n ((f_{101,i}^{(1l)} z_1 + f_{011,i}^{(1l)} z_2) w_i(0) + (f_{101,i}^{(1r)} z_1 + f_{011,i}^{(1r)} z_2) w_i(-1)) \\ \sum_{i=1}^n ((f_{101,i}^{(2l)} z_1 + f_{011,i}^{(2l)} z_2) w_i(0) + (f_{101,i}^{(2r)} z_1 + f_{011,i}^{(2r)} z_2) w_i(-1)) \\ \frac{1}{2} \begin{cases} \sum_{j,k \geq 0}^{j+k=2} f_{jk0}^{(3-)} z_1^j z_2^k, & \text{if } -1 \leq \vartheta < 0, \\ \sum_{j,k \geq 0}^{j+k=2} (f_{jk0}^{(3)} + f_{jk0}^{(3-)}) z_1^j z_2^k, & \text{if } \vartheta = 0 \end{cases} \end{bmatrix}. \quad (63) \end{aligned}$$

The subscripts of the constant coefficients $f_{jkm}^{(1,2)} \in \mathbb{R}$ in the first two equations and the vector ones $f_{jkm}^{(3)} \in \mathbb{R}^n$ in the third equation refer to the corresponding j th, k th, and m th orders of z_1 , z_2 , and w , respectively. The terms with the coefficients $f_{jk0}^{(3s)}$, $f_{jk0}^{(3c)}$ come from the linear combinations of $s_1(\vartheta)$ and $s_2(\vartheta)$. Note that all coefficients of the nonlinear terms are influenced by the scalar parameters $q_{1,2}$ (see (61)) related to the translational symmetry, except for $f_{jk0}^{(3)}$ and $f_{jk0}^{(3-)}$ (see (62)). The terms with coefficients $f_{jk0}^{(3)}$ and $f_{jk0}^{(3-)}$ refer to the structure of the modified nonlinear operator \mathcal{F}^- (see (48), (49)), that is, the vectors $f_{jk0}^{(3-)}$ appear due to the translational symmetry only, while the vectors $f_{jk0}^{(3)}$ would appear anyway.

The plane spanned by the eigenvectors s_1 and s_2 is tangent to the center-manifold (CM) at the origin. This means that the CM can be approximated locally as a truncated power series of w depending on the second order of the coordinates z_1 and z_2 :

$$w(\vartheta) = \frac{1}{2} (h_{20}(\vartheta) z_1^2 + 2h_{11}(\vartheta) z_1 z_2 + h_{02}(\vartheta) z_2^2). \quad (64)$$

The unknown coefficients h_{20} , h_{11} , and $h_{02} \in \mathbb{X}_{\mathbb{R}^n}$ can be determined by calculating the derivative of w in (64). On the one hand, it is expressed to the second order by the substitution of the linear part of the first two equations of (63):

$$\dot{w}(\vartheta) = -\omega h_{11}(\vartheta) z_1^2 + \omega (h_{20}(\vartheta) - h_{02}(\vartheta)) z_1 z_2 + \omega h_{11}(\vartheta) z_2^2. \quad (65)$$

On the other hand, this derivative can also be expressed by the third equation of (63). The comparison of the coefficients of z_1^2 , $z_1 z_2$, and z_2^2 gives a linear boundary value problem for the unknown coefficients of the CM, where the differential equation is

$$\begin{aligned} \begin{bmatrix} h'_{20}(\vartheta) \\ h'_{11}(\vartheta) \\ h'_{02}(\vartheta) \end{bmatrix} &= \begin{bmatrix} 0 & -2\omega\mathbf{I} & 0 \\ \omega\mathbf{I} & 0 & -\omega\mathbf{I} \\ 0 & 2\omega\mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} h_{20}(\vartheta) \\ h_{11}(\vartheta) \\ h_{02}(\vartheta) \end{bmatrix} \\ &- \begin{bmatrix} f_{200}^{(3c)} \\ \frac{1}{2}f_{110}^{(3c)} \\ f_{020}^{(3c)} \end{bmatrix} \cos(\omega\vartheta) - \begin{bmatrix} f_{200}^{(3s)} \\ \frac{1}{2}f_{110}^{(3s)} \\ f_{020}^{(3s)} \end{bmatrix} \sin(\omega\vartheta) - \begin{bmatrix} f_{200}^{(3-)} \\ \frac{1}{2}f_{110}^{(3-)} \\ f_{020}^{(3-)} \end{bmatrix}, \end{aligned} \quad (66)$$

and the boundary conditions can be written as

$$\begin{aligned} \begin{bmatrix} \mathbf{L} & 2\omega\mathbf{I} & 0 \\ -\omega\mathbf{I} & \mathbf{L} & \omega\mathbf{I} \\ 0 & -2\omega\mathbf{I} & \mathbf{L} \end{bmatrix} \begin{bmatrix} h_{20}(0) \\ h_{11}(0) \\ h_{02}(0) \end{bmatrix} + \begin{bmatrix} \mathbf{R} & 0 & 0 \\ 0 & \mathbf{R} & 0 \\ 0 & 0 & \mathbf{R} \end{bmatrix} \begin{bmatrix} h_{20}(-1) \\ h_{11}(-1) \\ h_{02}(-1) \end{bmatrix} \\ = - \begin{bmatrix} f_{200}^{(3c)} + f_{200}^{(3)} + f_{200}^{(3-)} \\ \frac{1}{2}(f_{110}^{(3c)} + f_{110}^{(3)} + f_{110}^{(3-)}) \\ f_{020}^{(3c)} + f_{020}^{(3)} + f_{020}^{(3-)} \end{bmatrix}. \end{aligned} \quad (67)$$

Note that the constant vector in the nonhomogeneous term of (66) formed from the vectors $f_{jk0}^{(3-)}$ does not show up if there is no translational symmetry, that is, if there is no zero characteristic exponent in the system. The general solution of (66) also contains extra terms that are related to the translational symmetry through the vectors $f_{jk0}^{(3-)}$:

$$\begin{aligned} \begin{bmatrix} h_{20}(\vartheta) \\ h_{11}(\vartheta) \\ h_{02}(\vartheta) \end{bmatrix} &= \begin{bmatrix} H_1 \\ H_2 \\ -H_1 \end{bmatrix} \cos(2\omega\vartheta) + \begin{bmatrix} -H_2 \\ H_1 \\ H_2 \end{bmatrix} \sin(2\omega\vartheta) + \begin{bmatrix} H_0 \\ 0 \\ H_0 \end{bmatrix} \\ &+ \frac{1}{3\omega} \left(\begin{bmatrix} f_{110}^{(3c)} + f_{200}^{(3s)} + 2f_{020}^{(3s)} \\ -\frac{1}{2}f_{110}^{(3s)} - f_{200}^{(3c)} + f_{020}^{(3c)} \\ -f_{110}^{(3c)} + 2f_{200}^{(3s)} + f_{020}^{(3s)} \end{bmatrix} \cos(\omega\vartheta) \right. \\ &\quad \left. + \begin{bmatrix} f_{110}^{(3s)} - f_{200}^{(3c)} - 2f_{020}^{(3c)} \\ \frac{1}{2}f_{110}^{(3c)} - f_{200}^{(3s)} + f_{020}^{(3s)} \\ -f_{110}^{(3s)} - 2f_{200}^{(3c)} - f_{020}^{(3c)} \end{bmatrix} \sin(\omega\vartheta) \right) \\ &- \frac{1}{4\omega} \begin{bmatrix} 0 \\ f_{200}^{(3-)} - f_{020}^{(3-)} \\ 2f_{110}^{(3-)} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} f_{200}^{(3-)} + f_{020}^{(3-)} \\ 0 \\ f_{200}^{(3-)} + f_{020}^{(3-)} \end{bmatrix} \vartheta. \end{aligned} \quad (68)$$

The unknown constant vectors H_0 , H_1 , and $H_2 \in \mathbb{R}^n$ are determined by the boundary conditions (67), which result in the linear nonhomogeneous equations

$$\begin{aligned} & \begin{bmatrix} \mathbf{L} + \mathbf{R} & 0 & 0 \\ 0 & \mathbf{L} + \mathbf{R} \cos(2\omega) & 2\omega\mathbf{I} + \mathbf{R} \sin(2\omega) \\ 0 & -(2\omega\mathbf{I} + \mathbf{R} \sin(2\omega)) & \mathbf{L} + \mathbf{R} \cos(2\omega) \end{bmatrix} \begin{bmatrix} H_0 \\ H_1 \\ H_2 \end{bmatrix} \\ &= \frac{1}{6\omega} \begin{bmatrix} (\mathbf{L} + \mathbf{R} \cos \omega)(-3f_{200}^{(3s)} - 3f_{020}^{(3s)}) + (\omega\mathbf{I} + \mathbf{R} \sin \omega)(-3f_{200}^{(3c)} - 3f_{020}^{(3c)}) \\ (\mathbf{L} + \mathbf{R} \cos \omega)(-2f_{110}^{(3c)} + f_{200}^{(3s)} - f_{020}^{(3s)}) + (\omega\mathbf{I} + \mathbf{R} \sin \omega)(2f_{110}^{(3s)} + f_{200}^{(3c)} - f_{020}^{(3c)}) \\ (\mathbf{L} + \mathbf{R} \cos \omega)(f_{110}^{(3s)} + 2f_{200}^{(3c)} - 2f_{020}^{(3c)}) + (\omega\mathbf{I} + \mathbf{R} \sin \omega)(f_{110}^{(3c)} - 2f_{200}^{(3s)} + 2f_{020}^{(3s)}) \end{bmatrix} \\ & - \frac{1}{4\omega} \begin{bmatrix} 2\omega(f_{200}^{(3)} + f_{020}^{(3)}) + 2\omega(\mathbf{I} + \mathbf{R})(f_{200}^{(3-)} + f_{020}^{(3-)}) - (\mathbf{L} + \mathbf{R})f_{110}^{(3-)} \\ 2\omega(f_{200}^{(3)} - f_{020}^{(3)}) + (\mathbf{L} + \mathbf{R})f_{110}^{(3-)} \\ 2\omega f_{110}^{(3)} - (\mathbf{L} + \mathbf{R})(f_{200}^{(3-)} - f_{020}^{(3-)}) \end{bmatrix}. \end{aligned} \quad (69)$$

Since $\mathbf{L} + \mathbf{R}$ is singular for systems with translational symmetry, the first (decoupled) group of nonhomogeneous equations for H_0 may look as though they are not solvable. However, the nonhomogeneous term on the right-hand side belongs to the image space of the coefficient matrix $\mathbf{L} + \mathbf{R}$, and this will result in a solution that is satisfactory for the CM calculation (see Section 5). Again, this issue is related to the translational symmetry in the system. If the reduction of the OpDE (28) were not carried out to the reduced OpDE (47) with respect to the relevant zero characteristic root, then the first (decoupled) group of (69) would lead to contradiction, and the CM calculation could not be continued.

The above calculation based on (64)–(69) is called the *center-manifold reduction*.

4.4. Poincaré Normal Form

Having the solution of (69), we can reconstruct the approximate equation of the CM via (68) and (64). Then calculating only the components $w(0)$ and $w(-1)$, and substituting them into the first two scalar equations of (63), we obtain the following equations that describe the flow restricted onto the two-dimensional CM:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \sum_{j,k \geq 0}^{j+k=2,3} g_{jk}^{(1)} z_1^j z_2^k \\ \sum_{j,k \geq 0}^{j+k=2,3} g_{jk}^{(2)} z_1^j z_2^k \end{bmatrix}. \quad (70)$$

We note that the coefficients of the second-order terms in the first two equations of (63) are not changed by the CM reduction, i.e., $f_{jk0}^{(1,2)} = g_{jk}^{(1,2)}$ when $j + k = 2$. The so-called Poincaré-Lyapunov constant in the Poincaré normal form of (70) can be determined by the Bautin formula

$$\begin{aligned} \Delta &= \frac{1}{8} \left(\frac{1}{\omega} \left((g_{20}^{(1)} + g_{02}^{(1)})(-g_{11}^{(1)} + g_{20}^{(2)} - g_{02}^{(2)}) + (g_{20}^{(2)} + g_{02}^{(2)})(g_{20}^{(1)} - g_{02}^{(1)} + g_{11}^{(2)}) \right) \right. \\ & \quad \left. + (3g_{30}^{(1)} + g_{12}^{(1)} + g_{21}^{(2)} + 3g_{03}^{(2)}) \right) \end{aligned} \quad (71)$$

(see [18]). It shows the type of bifurcation and approximate amplitude of the limit-cycle. The bifurcation is supercritical (subcritical) if $\Delta < 0$ ($\Delta > 0$), and the amplitude of the stable (unstable) oscillation is expressed by

$$A = \sqrt{-\frac{1}{\Delta} \operatorname{Re} \frac{d\lambda_{1,2}(\tau_{\text{cr}})}{d\tau} (\tau - \tau_{\text{cr}})}. \quad (72)$$

We note that the following formulas are valid with and without tildes, since they include only the frequency and the time in the form of the product $\omega t = \tilde{\omega} \tilde{t}$. Thus the first Fourier term of the oscillation on the center-manifold is

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = A \begin{bmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{bmatrix}. \quad (73)$$

Since not too far from the critical bifurcation (delay) parameter $x_t(\vartheta) \approx z_1(t)s_1(\vartheta) + z_2(t)s_2(\vartheta)$, and $x(t) = x_t(0)$ by definition, the formula (73) of the limit-cycle yields

$$\begin{aligned} x(t) &\approx z_1(t)s_1(0) + z_2(t)s_2(0) \\ &= A(s_1(0) \cos(\omega t) - s_2(0) \sin(\omega t)) \\ &= A(S_1 \cos(\omega t) - S_2 \sin(\omega t)). \end{aligned} \quad (74)$$

5. Application

As an illustration of the calculations above, let us consider a simple delayed car-following model. The cars follow each other on a circular road, i.e., we consider periodic boundary conditions. While real traffic systems are usually considered to be open, there are cases when highway rings around large cities, or city trams along closed loopleftike lanes, require models with real circular paths. Also, it is easier to carry out analytical investigations on these models, which may describe well the dynamics on portions of open road systems, too.

While our model is similar to that of Bando et al. [2], and a special case of the generalized braking force model of Helbing and Tilch [10], it takes into account an important delay effect, too. Bando et al. also included time delay in their latest model [1], which was recently investigated by Davis [4]. They only carried out analytical linear stability investigations and checked the global behavior by simulation when the equilibrium is linearly unstable. We instead investigate analytically the nonlinear behavior of the system but consider only the simplest case of two cars on a ring. A traffic model with two cars is oversimplified, of course, but important qualitative properties can be captured with this model, and the calculations can also be generalized along the algorithm of the above sections.

We consider the vehicles in the system with the same characteristics along a closed ring of length L , as sketched in Figure 1. The positions of the cars are denoted by y_1 and y_2 and their speeds by \dot{y}_1 and \dot{y}_2 . The governing equations of the vehicles' motion are the DDEs

$$\begin{aligned} \ddot{y}_1(t) &= \frac{v^0 - \dot{y}_1(t)}{T} + B(y_2(t - \tau) - y_1(t - \tau)), \\ \ddot{y}_2(t) &= \frac{v^0 - \dot{y}_2(t)}{T} + B(y_1(t - \tau) - y_2(t - \tau) + L). \end{aligned} \quad (75)$$

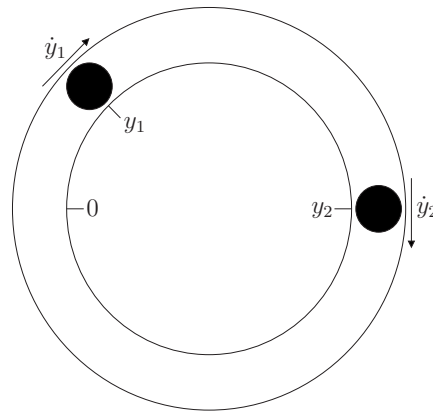


Fig. 1. Two cars on a ring with their positions and speeds.

Without the *braking force/function* B , the speed of the cars tends to the *desired speed* v^0 exponentially with the *relaxation time* T . The braking function B depends only on the distances of the vehicles, which is called *headway* and denoted by h . This is either $y_2 - y_1$ or $y_1 - y_2 + L$. The distances of the cars include the *reaction/reflex time* τ of the drivers in their arguments. As Davis [4] did, but in contrast with Bando et al. [1], we put the delay only into the braking term: The drivers immediately know their speeds; thus, the delay occurs in the interaction terms only.

The continuous function B depicted in Figure 2 is negative everywhere since it corresponds to deceleration. Its derivative is positive, because a driver pushes the brake pedal harder, if the car ahead is closer. However, this is valid while the car ahead is far enough and the driver does not want to stop, i.e., when $h \geq h_{\text{stop}}$. In contrast, the drivers decide to get to a full stop when they are too close to the car ahead, that is, when $0 < h < h_{\text{stop}}$. In

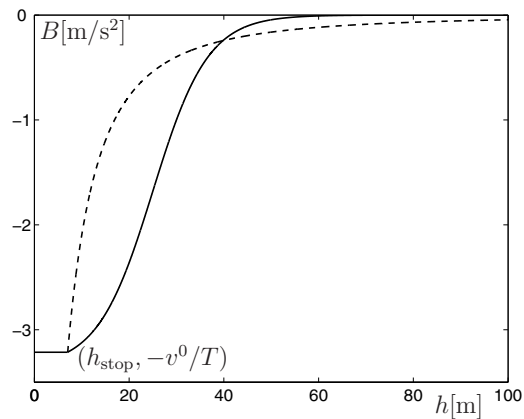


Fig. 2. Two drivers' braking functions with stopping region (the coordinates of the nonsmooth stopping point are displayed).

this case, the dynamics of the vehicles is simpler: $\ddot{y}_i(t) = -\dot{y}_i(t)/T$ for $i = 1, 2$, which corresponds to $B(h) \equiv -v^0/T$ in (75). Function B is also continuous at $h = h_{\text{stop}}$, but nothing can ensure the continuity of its first derivative there. Physically the nonsmooth first derivative seems correct, because it separates well the drivers' determination to move or stop.

The stationary motion of the vehicles (a kind of equilibrium of the system) can be described as

$$y_i^{\text{eq}}(t) = Vt + y_i^0, \quad (76)$$

for $i = 1, 2$, where

$$V = v^0 + TB(L/2) < v^0, \quad y_2^0 - y_1^0 = L/2. \quad (77)$$

The exact values of y_1^0 and y_2^0 are indeterminable because of the translational symmetry along the ring. Defining the perturbation

$$y_i^{\text{p}}(t) := y_i(t) - y_i^{\text{eq}}(t), \quad (78)$$

for $i = 1, 2$, and using Taylor series expansion about $h = L/2$ up to the third order of y_i^{p} , we can eliminate the zero-order terms. Thus, the equations for the perturbation terms assume the form

$$\begin{aligned} \ddot{y}_1^{\text{p}}(t) &= -\frac{\dot{y}_1^{\text{p}}(t)}{T} + \sum_{k=1}^3 b_k (y_2^{\text{p}}(t-\tau) - y_1^{\text{p}}(t-\tau))^k, \\ \ddot{y}_2^{\text{p}}(t) &= -\frac{\dot{y}_2^{\text{p}}(t)}{T} + \sum_{k=1}^3 b_k (y_1^{\text{p}}(t-\tau) - y_2^{\text{p}}(t-\tau))^k, \end{aligned} \quad (79)$$

where

$$b_1 = \frac{dB(L/2)}{dh}, \quad b_2 = \frac{1}{2} \frac{d^2B(L/2)}{dh^2}, \quad \text{and} \quad b_3 = \frac{1}{6} \frac{d^3B(L/2)}{dh^3}. \quad (80)$$

Note that it is possible to execute this expansion only for $L/2 \gg h_{\text{stop}}$, i.e., when the parameter $L/2$ is far from the nonsmooth point of the braking function B .

Let us introduce the dimensionless time $\tilde{t} := t/\tau$, which transforms the characteristic roots and the corresponding frequencies to $\tilde{\lambda} = \tau\lambda$ and $\tilde{\omega} = \tau\omega$, respectively, in the rescaled system. Introducing the new variables

$$x_i := \frac{d}{d\tilde{t}} y_i^{\text{p}} \quad x_{i+2} := y_i^{\text{p}}, \quad (81)$$

for $i = 1, 2$, and (by abuse of notation) dropping the tilde immediately, rewrite (79) in the form of rescaled first-order DDEs:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} &= \begin{bmatrix} -\tau/T & 0 & 0 & 0 \\ 0 & -\tau/T & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \tau^2 \begin{bmatrix} 0 & 0 & -b_1 & b_1 \\ 0 & 0 & b_1 & -b_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \\ x_4(t-1) \end{bmatrix} \\ &+ \tau^2 \begin{bmatrix} b_2(x_4(t-1) - x_3(t-1))^2 + b_3(x_4(t-1) - x_3(t-1))^3 \\ b_2(x_3(t-1) - x_4(t-1))^2 + b_3(x_3(t-1) - x_4(t-1))^3 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \quad (82)$$

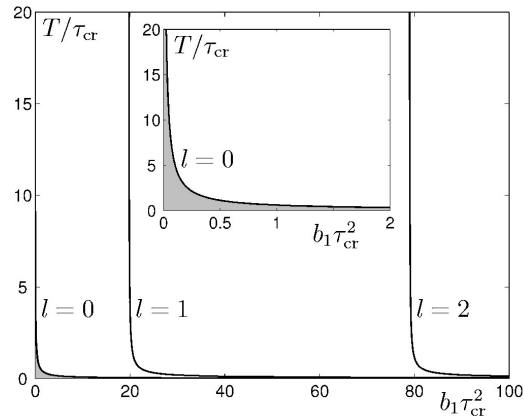


Fig. 3. D -curves in the plane of rescaled parameters for different values of l , where the enlarged section indicated at $l = 0$ is the actual stability boundary (stable region is grey).

The above equation belongs to the class of RFDE (1), where conditions (2), (4), and (5) are fulfilled. This can be shown by choosing Gx_t to be the linear part of the right-hand side of (82). Then the corresponding near-zero nonlinear part can be arranged in the form of F in (34) using

$$x_4(t-1) - x_3(t-1) = \frac{1}{\tau^2 b_1} \left((Gx_t)_1 + \frac{\tau}{T} (Gx_t)_3 \right). \quad (83)$$

The steady state $x(t) \equiv 0$ corresponds to the stationary motion of the original system. Considering the linear part of (82) and using the trial solution $x_i(t) = k_i e^{\lambda t}$, $k_i \in \mathbb{C}$, $i = 1, \dots, 4$, the characteristic equation is

$$D(\lambda, \tau) = (\lambda^2 + \tau\lambda/T + \tau^2 b_1 e^{-\lambda})^2 - (\tau^2 b_1 e^{-\lambda})^2 = 0. \quad (84)$$

Note that the state described by (76,77) can also be considered as a periodic motion because of the spatial periodicity along the ring with length L . However, this consideration does not change the analysis of the system: The coefficients of the linearized equations coming from (82) are constants; they do not depend on time despite of the periodic motion. The *continuous wave* states of semiconductor laser systems also possess the same feature, which is represented by a symmetry group and by graphical tools as well in [20].

Substituting the critical eigenvalue $\lambda = i\omega$ into (84) leads to equations

$$\cos \omega = \frac{\omega^2}{2b_1 \tau_{\text{cr}}^2}, \quad \sin \omega = \frac{\omega}{2b_1 T \tau_{\text{cr}}}, \quad (85)$$

which describe the D -curves in the plane of the rescaled parameters $b_1\tau_{\text{cr}}^2$ and T/τ_{cr} . The curves are parameterized by the rescaled frequency ω . The curves situated in the physically relevant parameter domain ($b_1\tau_{\text{cr}}^2 \geq 0, T/\tau_{\text{cr}} \geq 0$) are restricted to $2l\pi < \omega < (2l + 1/2)\pi, l \in \mathbb{N}$. The different values of l correspond to different curves, which do not cross each other, as displayed in Figure 3. Using an infinite-dimensional Routh-Hurwitz criterion like [18], the stability investigations show that the system is stable on the left side of the curve indicated by $l = 0$ (grey area in Figure 3) and unstable to the right. Crossing the curves “from left to right” means that a complex conjugate pair of characteristic roots goes to the right-hand side of the complex plane; hence the steady state never becomes stable again after losing its stability crossing the $l = 0$ curve. Thus the curve belonging to $l = 0$ is the only stability boundary.

Using formula (19), we calculate the necessary condition of Hopf bifurcation:

$$\operatorname{Re}\left(\frac{d\lambda_1(\tau_{\text{cr}})}{d\tau}\right) = \mathcal{E} \frac{1}{\tau_{\text{cr}}} \left(\frac{\tau_{\text{cr}}^2}{T^2} + 2\omega^2\right) > 0, \quad (86)$$

where

$$\mathcal{E} = \left(\left(\frac{\tau_{\text{cr}}}{T\omega} - \omega\right)^2 + \left(\frac{\tau_{\text{cr}}}{T} + 2\right)^2\right)^{-1}. \quad (87)$$

The positiveness of this quantity holds all along the stability boundary. The coefficient matrices

$$\mathbf{L} = \begin{bmatrix} -\tau_{\text{cr}}/T & 0 & 0 & 0 \\ 0 & -\tau_{\text{cr}}/T & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \tau_{\text{cr}}^2 \begin{bmatrix} 0 & 0 & -b_1 & b_1 \\ 0 & 0 & b_1 & -b_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (88)$$

of (82) show up in the linear operator \mathcal{A} (29), while the nonlinear terms of (82) define the nonlinear operator of the OpDE (28):

$$\mathcal{F}(\phi)(\vartheta) = \tau_{\text{cr}}^2 \begin{cases} 0, & \text{if } -1 \leq \vartheta < 0, \\ \begin{bmatrix} b_2(\phi_4(-1) - \phi_3(-1))^2 + b_3(\phi_4(-1) - \phi_3(-1))^3 \\ b_2(\phi_3(-1) - \phi_4(-1))^2 + b_3(\phi_3(-1) - \phi_4(-1))^3 \\ 0 \\ 0 \end{bmatrix}, & \text{if } \vartheta = 0, \end{cases} \quad (89)$$

through the vector $F(\phi(0), \phi(-1))$ shown for $\vartheta = 0$, according to (30). The constant coefficients of the eigenvectors of operators \mathcal{A} and \mathcal{A}^* belonging to the zero eigenvalue are

$$S_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad N_0 = \frac{1}{2} \begin{bmatrix} T/\tau_{\text{cr}} \\ T/\tau_{\text{cr}} \\ 1 \\ 1 \end{bmatrix}. \quad (90)$$

The nonlinear operator of the reduced OpDE (47) can be written into the form

$$\mathcal{F}^-(\phi)(\vartheta) = \tau_{\text{cr}}^2 \begin{cases} \begin{bmatrix} 0 \\ 0 \\ -(T/\tau_{\text{cr}})b_2(\phi_4(-1) - \phi_3(-1))^2 \\ -(T/\tau_{\text{cr}})b_2(\phi_3(-1) - \phi_4(-1))^2 \end{bmatrix}, & \text{if } -1 \leq \vartheta < 0, \\ \begin{bmatrix} b_2(\phi_4(-1) - \phi_3(-1))^2 + b_3(\phi_4(-1) - \phi_3(-1))^3 \\ b_2(\phi_3(-1) - \phi_4(-1))^2 + b_3(\phi_3(-1) - \phi_4(-1))^3 \\ -(T/\tau_{\text{cr}})b_2(\phi_4(-1) - \phi_3(-1))^2 \\ -(T/\tau_{\text{cr}})b_2(\phi_3(-1) - \phi_4(-1))^2 \end{bmatrix}, & \text{if } \vartheta = 0, \end{cases} \quad (91)$$

according to (48,49). It can be seen that third-order terms in ϕ are missing in the vector $-N_0^* \mathcal{F}(\phi)(0)S_0 = -N_0^* F(\phi(0), \phi(-1))S_0$ due to a special symmetry of the two-car system. This symmetry comes from $(\phi_4 - \phi_3) = -(\phi_3 - \phi_4)$, which does not hold for larger number of cars, of course. On the other hand, the operator \mathcal{F}^- appears in the third equations of (60), (62), where only second-order terms are needed, as mentioned there.

The constant coefficients of the critical eigenvectors of operator \mathcal{A} belonging to the critical eigenvalue $i\omega$ are

$$S_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad S_2 = - \begin{bmatrix} 0 \\ 0 \\ 1/\omega \\ -1/\omega \end{bmatrix}, \quad (92)$$

while the coefficient vectors of operator \mathcal{A}^* associated with eigenvalue $-i\omega$ are

$$N_1 = \mathcal{E} \begin{bmatrix} \tau_{\text{cr}}/T + 2 \\ -(\tau_{\text{cr}}/T + 2) \\ \tau_{\text{cr}}^2/T^2 + \tau_{\text{cr}}/T + \omega^2 \\ -(\tau_{\text{cr}}^2/T^2 + \tau_{\text{cr}}/T + \omega^2) \end{bmatrix}, \quad N_2 = -\mathcal{E} \begin{bmatrix} \tau_{\text{cr}}/(T\omega) - \omega \\ -(\tau_{\text{cr}}/(T\omega) - \omega) \\ \tau_{\text{cr}}^2/(T^2\omega) + 2\omega \\ -(\tau_{\text{cr}}^2/(T^2\omega) + 2\omega) \end{bmatrix}. \quad (93)$$

Here, $q_1 = q_2 = 0$, because $RS_0 = N_1^* S_0 = N_2^* S_0 = 0$ (see (61)). One may check that this holds for more than two cars as well. Hence the coefficients of the nonlinear terms in (63) are not changed by the translational symmetry; only the coefficient vectors $f_{jk0}^{(3-)}$ appear due to this symmetry.

The special two-car-symmetry $(\phi_4 - \phi_3) = -(\phi_3 - \phi_4)$ together with the zero values of $q_{1,2}$ result in the second-order terms in ϕ disappearing from $(N_j^* - q_j N_0^*) \mathcal{F}(\phi)(0) = (N_j^* - q_j N_0^*) F(\phi(0), \phi(-1))$ for $j = 1, 2$ in (62). Thus, one obtains $f_{jk0}^{(1)} = f_{jk0}^{(2)} = 0$ for $j + k = 2$ and $f_{jk1,i}^{(1l)} = f_{jk1,i}^{(2l)} = f_{jk1,i}^{(1r)} = f_{jk1,i}^{(2r)} = 0$ in the first two equations of (63), and $f_{jk0}^{(3c)} = f_{jk0}^{(3s)} = 0$ in the third equation of (63).

Finally, we obtain the equations (63) for z_1, z_2 , and w in the form

$$\begin{aligned} \frac{d}{dt} z_1 &= \omega z_2 + 2\mathcal{E} \frac{b_3 \omega^3}{b_1^3 \tau_{\text{cr}}^4} \left(2 + \frac{\tau_{\text{cr}}}{T} \right) \left(\frac{\tau_{\text{cr}}^3}{T^3 \omega^3} z_1^3 + 3 \frac{\tau_{\text{cr}}^2}{T^2 \omega^2} z_1^2 z_2 + 3 \frac{\tau_{\text{cr}}}{T \omega} z_1 z_2^2 + z_2^3 \right), \\ \frac{d}{dt} z_2 &= -\omega z_1 - 2\mathcal{E} \frac{b_3 \omega^3}{b_1^3 \tau_{\text{cr}}^4} \left(\frac{\tau_{\text{cr}}}{T \omega} - \omega \right) \left(\frac{\tau_{\text{cr}}^3}{T^3 \omega^3} z_1^3 + 3 \frac{\tau_{\text{cr}}^2}{T^2 \omega^2} z_1^2 z_2 + 3 \frac{\tau_{\text{cr}}}{T \omega} z_1 z_2^2 + z_2^3 \right), \end{aligned}$$

$$\frac{d}{dt}w(\vartheta) = \mathcal{A}w(\vartheta) + \frac{b_2\omega^2}{b_1^2\tau_{\text{cr}}^2} \left(\frac{\tau_{\text{cr}}^2}{T^2\omega^2}z_1^2 + 2\frac{\tau_{\text{cr}}}{T\omega}z_1z_2 + z_2^2 \right) \begin{cases} \begin{bmatrix} 0 \\ 0 \\ -T/\tau_{\text{cr}} \\ -T/\tau_{\text{cr}} \end{bmatrix}, & \text{if } -1 \leq \vartheta < 0, \\ \begin{bmatrix} 1 \\ 1 \\ -T/\tau_{\text{cr}} \\ -T/\tau_{\text{cr}} \end{bmatrix}, & \text{if } \vartheta = 0. \end{cases} \quad (94)$$

Thus, the first two scalar equations are already in the form of a system restricted to the center-manifold. For the sake of presenting the theory through this example, let us calculate the center-manifold. As we mentioned, due to two-car symmetry $f_{jk0}^{(3c)} = f_{jk0}^{(3s)} = 0$; nevertheless, the differential equation (66) is nonhomogeneous since the coefficients $f_{jk0}^{(3-)}$ are nonzero, due to the singularity related to the translational symmetry. Thus, (69) gives the two decoupled equations

$$(\mathbf{L} + \mathbf{R})H_0 = -\frac{b_2\omega^2}{b_1^2\tau_{\text{cr}}^2} \left(\frac{\tau_{\text{cr}}^2}{T^2\omega^2} + 1 \right) \begin{bmatrix} 1 \\ 1 \\ -T/\tau_{\text{cr}} \\ -T/\tau_{\text{cr}} \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{L} + \mathbf{R} \cos(2\omega) & 2\omega\mathbf{I} + \mathbf{R} \sin(2\omega) \\ -(2\omega\mathbf{I} + \mathbf{R} \sin(2\omega)) & \mathbf{L} + \mathbf{R} \cos(2\omega) \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = -\frac{b_2\omega^2}{b_1^2\tau_{\text{cr}}^2} \begin{bmatrix} \tau_{\text{cr}}^2/(T^2\omega^2) - 1 \\ \tau_{\text{cr}}^2/(T^2\omega^2) - 1 \\ 0 \\ 0 \\ 2\tau_{\text{cr}}/(T\omega) \\ 2\tau_{\text{cr}}/(T\omega) \\ 0 \\ 0 \end{bmatrix}. \quad (95)$$

The second equation of (95) would be the same without elimination of the zero eigendirection, because

$$(\mathbf{L} + \mathbf{R})f_{110}^{(3-)} = (\mathbf{L} + \mathbf{R})(f_{200}^{(3-)} - f_{020}^{(3-)}) = 0 \quad (96)$$

(see (69)). It follows from the facts that $(\mathbf{L} + \mathbf{R})\text{col}[0, 0, 1, 1] = 0$ and that the vectors $f_{110}^{(3-)}$, $f_{200}^{(3-)}$, and $f_{020}^{(3-)}$ are proportional to $\text{col}[0, 0, 1, 1]$. The first two coordinates of the vectors H_1 and H_2 can be determined, but these will not be important later. Their third and fourth coordinates are undetermined, but their differences can be computed: $H_{1,4} - H_{1,3} = 0$ and $H_{2,4} - H_{2,3} = 0$. It is satisfactory to use this result, because everything depends on the differences of these components in all equations.

The first equation of (95) is different from the form obtained without the elimination of the translational-symmetry-related singularity, because

$$(\mathbf{I} + \mathbf{R})(f_{200}^{(3-)} - f_{020}^{(3-)}) = f_{200}^{(3-)} - f_{020}^{(3-)} \neq 0, \quad (97)$$

while $(\mathbf{L} + \mathbf{R})f_{110}^{(3-)} = 0$ again (see (69)). Here, $H_{0,1}$ and $H_{0,2}$ are determined, while $H_{0,3}$ and $H_{0,4}$ are not, but $H_{0,4} - H_{0,3} = 0$, which is a satisfactory solution, again. Note that

without the reduction of the OpDE, the third and fourth coordinates of the right-hand side in the first equation of (95) would be zero, which would lead to contradiction for $H_{0,3}$ and $H_{0,4}$. This is the main reason for the elimination of the translational-symmetry-related singularity in the Hopf bifurcation calculation.

Consequently, we get the result $w_4(\vartheta) - w_3(\vartheta) = 0$, which corresponds to the fact that the center-manifold reduction is not necessary in this special two-car case. Note that, for more than two cars, the above center-manifold reduction is necessary.

Using formula (71) we can calculate the quantity Δ from (94):

$$\Delta = \varepsilon \frac{3b_3}{4b_1^3\tau_{cr}^4} \left(\frac{\tau_{cr}^2}{T^2} + \omega^2 \right) \left(\frac{\tau_{cr}^2}{T^2} + \frac{\tau_{cr}}{T} + \omega^2 \right). \quad (98)$$

All the quantities are positive in Δ except b_3 , which determines the sign of Δ . Thus, the bifurcation is supercritical in the case of $\Delta \sim b_3 < 0$ and subcritical in the case of $\Delta \sim b_3 > 0$. The dynamics of the cars is essentially different in these two cases: The bifurcating periodic motion is orbitally stable or unstable, respectively.

In the subcritical case, simulations show that a stable periodic solution coexists with the unstable limit cycle bifurcated from the stable steady-state. The existence of this motion is also confirmed by continuation studies in [13]. Its amplitude is larger than the amplitude of the unstable limit cycle, and the cars stop (or nearly stop) during this oscillation. It is called *stop-and-go* (or *slow-and-go*) motion in traffic dynamics and corresponds to the constant section of the braking function B shown in Figure 2. The dynamics of the system is switching between the moving and stopping motion corresponding to the discontinuity of the first derivative of the braking function B . We have proven that the sign of the third derivative of the function B determines the type of Hopf bifurcation. This can change in the case of a larger number of cars: The sign of the second derivative of the braking function becomes important, too.

From (86) and (98), we can calculate the amplitude of the arising oscillation with the help of formula (72). The overall oscillation of the vehicles is described by

$$\begin{bmatrix} y_1^p(t) \\ y_2^p(t) \end{bmatrix} = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin(\omega t), \quad (99)$$

where the amplitude has the form

$$A = \sqrt{-\frac{b_1}{3b_3} \frac{4b_1^2\tau_{cr}^4/\omega^2 + \omega^2}{4b_1^2\tau_{cr}^4/\omega^2 + \tau_{cr}/T} \left(\frac{\tau}{\tau_{cr}} - 1 \right)}. \quad (100)$$

Here, we used

$$\frac{4b_1^2\tau_{cr}^4}{\omega^2} = \left(\frac{\tau_{cr}^2}{T^2} + \omega^2 \right), \quad (101)$$

originated in (85), from which the frequency ω can also be determined as a function of the parameters $b_1\tau_{cr}^2$ and T/τ_{cr} .

As mentioned above, it is possible to choose other bifurcation parameters, for example, the average distance $h^* := L/2$ of cars. In this case, we can check the Hopf condition by computing the quantity

$$\operatorname{Re} \left(\frac{d\lambda_1(h_{cr}^*)}{dh^*} \right) = \varepsilon \frac{2b_{2cr}}{b_{1cr}} \left(\frac{\tau^2}{T^2} + \frac{\tau}{T} + \omega^2 \right), \quad (102)$$

and thus (72) gives the vibration amplitude

$$A = \sqrt{-\frac{2b_{2\text{cr}}}{3b_{3\text{cr}}}(h^* - h_{\text{cr}}^*)}, \quad (103)$$

where the derivatives b_k , $k = 1, 2, 3$ (80) take the values $b_{k\text{cr}}$, $k = 1, 2, 3$, respectively, at the critical point $L/2 = h^* = h_{\text{cr}}^*$. This simple amplitude formula is fully determined by the braking function only.

6. Conclusion

We have given an algorithm for the Hopf bifurcation calculation including a center-manifold reduction in infinite-dimension for time-delayed systems with translational symmetry. In these systems a relevant zero characteristic exponent exists for *any* values of the chosen bifurcation parameter (which was the time delay in our study). The CM reduction related to the pure imaginary characteristic roots cannot be carried out by the standard algorithms used in the literature [3], [9], [18]: A linear nonhomogeneous equation occurs with singular coefficient matrix leading to contradiction. The central idea of our method lies in the projection of the OpDE form of the system onto the complement of the eigenspace related to the relevant zero eigenvalue. The Hopf bifurcation calculation is presented then on the reduced OpDE.

The method was applied to a model of a delayed car-following system. Cars following each other along a closed ring represent a system with translational symmetry, while they also exhibit self-excited oscillations originated in a Hopf bifurcation. Typically, unstable periodic vibrations arise around the stable stationary traffic flow. If this stationary motion is “more strongly” perturbed than the unstable limit-cycle, then a stable periodic stop-/slow-and-go motion occurs as a global attractor representing a traffic jam traveling backwards along the ring. These results are proven analytically for two cars and checked by simulations for several cars. Further analysis with continuation methods is in progress [13].

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