Normal form calculations for day-to-day traffic dynamics

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Summary. The effects of driver experience delay on the dynamics of a nonlinear day-to-day traffic model are investigated. When parameters are varied oscillations may arise via flip and Neimark-Sacker bifurcations in the non-delayed and delayed cases, respectively. The nonlinear behavior is analyzed using normal form calculations which are validated by numerical continuation and simulation. It is shown that the delay can significantly increase the period of the arising oscillations and may lead to extended domains of bistability where the user equilibrium coexist with stable and unstable oscillations.

Introduction

Day-to-day traffic assignment models describe how users forecast travel conditions according to their own experience and information from other sources, and how they make their route decisions, either by repeating their previous choice or re-routing according their forecast. Many travelers execute the same trip in every few days, not daily, which leads to time delays that can significantly influence the arising flow patters in transportation networks. Here we formulate this problem at the population level using discrete-time dynamical systems. Nonlinearities appear due driver decision making and cost evaluation while incorporating the average driver experience delay increases the dimension of the state space. By applying rigorous analysis we show that the delay changes the bifurcation structure of the system significantly, resulting in Neimark-Sacker bifurcations as opposed to flip bifurcations that occur in the non-delayed case. Linear stability analysis predicts, that the size of the linearly unstable parameter domain as well as the period of the arising oscillations increase with the delay. Moreover, delays and nonlinearities result in an extended domain of bistability where the equilibrium coexists with stable and unstable oscillations. A more general treatment can be found in [9] with less technical details presented.

Day-to-day traffic model for a two-route network

In this paper we consider the two-route network shown in Fig. 1(a) where the origin (O) and the destination (D) are linked by two arcs. To describe the time evolution of the cost and the flow we propose the dynamic model

$$\begin{aligned} \mathbf{c}_{t+1} &= \alpha\, C(\mathbf{f}_{t-\tau}) + (1-\alpha)\,\mathbf{c}_t\,, \\ \mathbf{f}_{t+1} &= \beta\, F(\mathbf{c}_{t+1}) + (1-\beta)\,\mathbf{f}_t\,. \end{aligned} \tag{1}$$

The first equation updates of the cost vector $\mathbf{c} = [c_1, c_2]^{\mathrm{T}}$ and describes how users forecast costs based on their experiences $\tau+1$ days ago. Here τ represents the *average driver experience delay* while the parameter $\alpha \in [0,1]$ is the weight attributed to the forecasted costs against yesterday's actual costs. The second equation updates the flow vector $\mathbf{f} = [f_1, f_2]^{\mathrm{T}}$ according to how many users decide to re-route and how many are inertial. The parameter $\beta \in [0,1]$ is the weight attributed to reconsidering the previous choice. For $\tau = 0$ Eq. (1) recovers the non-delayed model in [2].
Model (1) contains two nonlinear functions, the *arc cost function* and the *network loading function*:

$$C(\mathbf{f}) = \begin{bmatrix} C_1(f_1,f_2) \\ C_2(f_1,f_2) \end{bmatrix} = \begin{bmatrix} l_0\big(1+l_1(f_1/f_{\mathrm{c}})^4\big) \\ l_0\big(1+l_1(f_2/f_{\mathrm{c}})^4\big) \end{bmatrix}, \qquad F(\mathbf{c}) = \begin{bmatrix} F_1(c_1,c_2) \\ F_2(c_1,c_2) \end{bmatrix} = \begin{bmatrix} \frac{d}{1+\exp\big(-(c_2-c_1)/\theta\big)} \\ \frac{d}{1+\exp\big(-(c_1-c_2)/\theta\big)} \end{bmatrix}, \tag{2}$$

which are plotted in Fig. 1(b,c). The arc cost function $C(\mathbf{f})$ describes how the arc costs depend on the arc flows and here we use the so-called Bureau of Public Roads (BPR) function [1] where l_0 is the zero-flow cost, $l_0 l_1$ is the cost range, and f_{c} is the arc capacity, as shown in Fig. 1(b). The network loading function $F(\mathbf{c})$ establishes relations between the arc costs and the arc flows by describing how the demand d is assigned on the arcs. The logit-type function used here is commonly used to describe human decision making; see Fig. 1(c). The distance between the two inflection points is proportional to the parameter θ that represents the characteristic price difference needed to switch the route. One may show that

$$f_1^* = f_2^* = d/2\,, \qquad c_1^* = c_2^* = l_0\big(1 + l_1\big(\tfrac{d}{2f_{\mathrm{c}}}\big)^4\big) \tag{3}$$

is a unique fixed point of (1,2) which is often called the user equilibrium or Nash equilibrium [7].
In order to obtain the state space representation of system (1), one needs to use $2(2+\tau)$ scalar variables that are contained by the vectors $\mathbf{c}_t,\ \mathbf{f}_t,\ \mathbf{f}_{t-1},\ \cdots,\ \mathbf{f}_{t-\tau}$. However, exploiting the flow conservation constraint $f_1 + f_2 = d$ and the fact that the network loading function only depends on the cost difference

$$\Delta c = c_1 - c_2\,, \tag{4}$$

the dimension of the state space reduces to $2+\tau$. Thus, using Taylor expansion one can obtain the third-order approximation of (1,2) in the form

$$\begin{aligned} \Delta c_{t+1} &= \frac{\alpha l_0 l_1}{f_c^4}\left(4\,d\,f_{1,t-\tau}^3 - 6\,d^2 f_{1,t-\tau}^2 + 4d^3 f_{1,t-\tau} - d^4\right) + (1-\alpha)\Delta c_t\,, \\ f_{1,t+1} &= \frac{\beta d}{2} - \frac{\beta d}{4\theta}\Delta c_{t+1} + \frac{\beta d}{48\theta^3}\Delta c_{t+1}^3 + (1-\beta) f_{1,t}\,. \end{aligned} \tag{5}$$

In the forthcoming two sections we focus on the simple cases $\tau = 0$ and $\tau = 1$ and analyze the bifurcations arising in (5) with the help of normal form calculations while using β as the bifurcation parameter. This will be followed by numerical investigation of the original system (1,2).

Flip bifurcation in the non-delayed system

In this section we consider $\tau = 0$ and investigate the flip (period doubling) bifurcations arising in the system. Let us define the perturbation

$$\Delta f = f_1 - d/2\,. \tag{6}$$

For $\tau = 0$ we can rewrite (5) as

$$\begin{bmatrix}\Delta c_{t+1}\\ \Delta f_{t+1}\end{bmatrix} = \begin{bmatrix} 1-\alpha & \frac{8\alpha\theta q}{d} \\ -(1-\alpha)\frac{\beta d}{4\theta} & 1-\beta(1+2\alpha q)\end{bmatrix}\begin{bmatrix}\Delta c_t\\ \Delta f_t\end{bmatrix} + \begin{bmatrix} \frac{32\alpha\theta q}{d^3}\Delta f_t^3 \\ -\frac{8\alpha\beta q}{d^2}\Delta f_t^3 + \frac{\beta d}{48\theta^3}\left(\frac{8\alpha\theta q}{d}\Delta f_t + (1-\alpha)\Delta c_t\right)^3\end{bmatrix}. \tag{7}$$

where

$$q = \frac{l_0 l_1 d^4}{8\theta f_c^4} \geq 0\,, \tag{8}$$

is a dimensionless parameter. The characteristic equation becomes

$$(\lambda + \alpha - 1)(\lambda + \beta - 1) + 2\alpha\beta q\lambda = 0\,. \tag{9}$$

The corresponding eigenvalues are

$$\lambda_{1,2}(\beta) = \frac{1}{2}\left(2 - \alpha - \beta - 2\alpha\beta q \mp \sqrt{(\alpha-\beta)^2 - 4\alpha\beta q(2-\alpha-\beta-\alpha\beta q)}\right), \tag{10}$$

and using these in the linear part of (7) we obtain the eigenvectors

$$\mathbf{s}_{1,2}(\beta) = \begin{bmatrix} \frac{2\theta}{(1-\alpha)\beta d}\left(\alpha - \beta - 2\alpha\beta q \pm \sqrt{(\alpha-\beta)^2 - 4\alpha\beta q(2-\alpha-\beta-\alpha\beta q)}\right) \\ 1 \end{bmatrix}. \tag{11}$$

By investigating (9,10) one can show that the system loses stability via flip (period doubling) bifurcation at

$$\beta_{\mathrm{F}} = \frac{4-2\alpha}{2-\alpha(1-2q)}\,, \tag{12}$$

where we have $\lambda_1(\beta_{\mathrm{F}}) = -1$ while $-1 < \lambda_2(\beta_{\mathrm{F}}) < 1$. By taking the derivative of the characteristic equation (9) at the critical point (12) we obtain

$$\left.\frac{\mathrm{d}\lambda_1}{\mathrm{d}\beta}\right|_{\beta_{\mathrm{F}}} = -\frac{\left[2-\alpha(1-2q)\right]^2}{4(1-\alpha)+\alpha^2(1+2q)} < 0\,, \tag{13}$$

The negative sign indicates that increasing β makes the critical eigenvalue move through -1 from left to right leading to a stability loss. This can be observed in Fig. 1(d) for $d = 1, q = 1, \alpha = 1/2$, where the horizontal solid line indicates stable equilibrium while the horizontal dashed line denotes unstable equilibrium. In this case, (12) gives $\beta_{\mathrm{F}} = 6/5$ as marked by a cross.

Using the matrix

$$\mathbf{T}(\beta) = \begin{bmatrix}\mathbf{s}_1(\beta) & \mathbf{s}_2(\beta)\end{bmatrix}\,, \tag{14}$$

we can define the coordinate transformation

$$\begin{bmatrix}\Delta c\\ \Delta f\end{bmatrix} = \mathbf{T}(\beta)\begin{bmatrix}\xi\\ \eta\end{bmatrix}. \tag{15}$$

Substituting (15) into (7) we obtain the form

$$\begin{bmatrix}\xi_{t+1}\\ \eta_{t+1}\end{bmatrix} = \begin{bmatrix}\lambda_1(\beta) & 0\\ 0 & \lambda_2(\beta)\end{bmatrix}\begin{bmatrix}\xi_t\\ \eta_t\end{bmatrix} + \begin{bmatrix} a_{30}(\beta)\xi_t^3 + a_{21}(\beta)\xi_t^2\eta_t + a_{12}(\beta)\xi_t\eta_t^2 + a_{03}(\beta)\eta_t^3 \\ \cdots \end{bmatrix}. \tag{16}$$

Since the first equation only contains third-order terms, no center manifold reductions is necessary to obtain the normal form and the second equation can be neglected. In fact, the lack of second-order terms also means that the first equation in (16) is equivalent to the normal form of the flip bifurcation

$$\xi_{t+1} = \lambda_1(\beta)\xi_t - \delta(\beta)\xi_t^3\,, \tag{17}$$

that describes the motion in the center manifold [5]. (The case with second-order terms included is explained in the Appendix and in [8].) In the vicinity of the bifurcation point, that is, for $\beta \approx \beta_{\rm F}$, the oscillations can be written as

$$\xi_t = \sqrt{\frac{\frac{\mathrm{d}\lambda_1}{\mathrm{d}\beta}\big|_{\beta_{\rm F}}}{\delta(\beta_{\rm F})}(\beta-\beta_{\rm F})}\cdot(-1)^t\,. \tag{18}$$

Moreover, the criticality of the bifurcation is given by

$$\delta(\beta_{\rm F}) = -a_{30}(\beta_{\rm F}) = -\frac{16\alpha q\big[12(1-\alpha)+\alpha^2(3-4q^2)\big]}{3d^2(\alpha-2)\big[4(1-\alpha)+\alpha^2(1+2q)\big]}\,. \tag{19}$$

so that

$$\begin{cases}\delta(\beta_{\rm F})>0 & \text{if} \quad 0<\alpha<\frac{2\sqrt{3}}{2q+\sqrt{3}} \quad \Rightarrow \text{ subcritical flip}\,,\\ \delta(\beta_{\rm F})<0 & \text{if} \quad \frac{2\sqrt{3}}{2q+\sqrt{3}}<\alpha<2 \quad \Rightarrow \text{ supercritical flip}\,.\end{cases} \tag{20}$$

In the subcritical case unstable oscillations appear where the equilibrium is stable, while in the supercritical case stable oscillations arise where the equilibrium is unstable. Applying the transformation (15) at $\beta=\beta_{\rm F}$ and restricting the system to the center manifold $\eta=0$, we have $\Delta f \approx T_{21}(\beta_{\rm F})\xi=\xi$. Consequently, (6,13,18,19) give the oscillations

$$\Delta f_t = \sqrt{S_{\rm F}\big(\beta-\beta_{\rm F}\big)}\cdot(-1)^t \quad \Rightarrow \quad f_{1,t} = d/2+\sqrt{S_{\rm F}\big(\beta-\beta_{\rm F}\big)}\cdot(-1)^t\,, \tag{21}$$

where

$$S_{\rm F} = \frac{3d^2(\alpha-2)\big[2-\alpha(1-2q)\big]^2}{16\alpha q\big[12(1-\alpha)+\alpha^2(3-4q^2)\big]}\,. \tag{22}$$

The minima and maxima of the arising period-two unstable oscillations are plotted as black dashed curves in Fig. 1(d) for $d=1, q=1, \alpha=1/2$. Since $\alpha<4\sqrt(3)-6$, according to (20) the bifurcation is subcritical and indeed (19) yields $\delta(\beta_{\rm F})=368/99>0$ while (22) results in $S_{\rm F}=-225/386<0$.

Neimark-Sacker bifurcation in the delayed system

Here we consider $\tau=1$ and analyze the oscillations arising through Neimark-Sacker bifurcation. In additional to Δc_t and Δf_t (cf. (4) and (6)), we need to define the state variable

$$\Delta g_t = \Delta f_{t-1}\,, \tag{23}$$

which appears in the right hand side of (5). Thus, we obtain the system

$$\begin{bmatrix}\Delta c_{t+1}\\ \Delta f_{t+1}\\ \Delta g_{t+1}\end{bmatrix} = \begin{bmatrix}1-\alpha & 0 & \frac{8\alpha\theta q}{d}\\ -(1-\alpha)\frac{\beta d}{4\theta} & 1-\beta & -2\alpha\beta q\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta c_t\\ \Delta f_t\\ \Delta g_t\end{bmatrix} + \begin{bmatrix}\frac{32\alpha\theta q}{d^3}\Delta g_t^3\\ -\frac{8\alpha\beta q}{d^2}\Delta g_t^3+\frac{\beta d}{48\theta^3}\Big(\frac{8\alpha\theta q}{d}\Delta g_t+(1-\alpha)\Delta c_t\Big)^3\\ 0\end{bmatrix}\,, \tag{24}$$

cf. (7). The characteristic equation becomes

$$(\lambda+\alpha-1)(\lambda+\beta-1)\lambda+2\alpha\beta q\lambda = 0\,, \tag{25}$$

that gives the eigenvalues

$$\lambda_{1,2}(\beta) = \frac{1}{2}\Big(2-\alpha-\beta\pm\mathrm{i}\sqrt{8\alpha\beta q-(\alpha-\beta)^2}\Big)\,, \qquad \lambda_3 = 0\,. \tag{26}$$

and the linear part of (24) gives the eigenvectors

$$\mathbf{s}_{1,2}(\beta) = \begin{bmatrix}\frac{2\theta}{\beta d}\big(\alpha-\beta\mp\mathrm{i}\sqrt{8\alpha\beta q-(\alpha-\beta)^2}\big)\\ \frac{1}{2}\big(2-\alpha-\beta\pm\mathrm{i}\sqrt{8\alpha\beta q-(\alpha-\beta)^2}\big)\\ 1\end{bmatrix}\,, \qquad \mathbf{s}_3 = \begin{bmatrix}\frac{8\alpha\theta q}{d(\alpha-1)}\\ 0\\ 1\end{bmatrix}\,. \tag{27}$$

Calculating

$$|\lambda_{1,2}(\beta)| = \sqrt{(1-\alpha)(1-\beta)+2\alpha\beta q}\,, \tag{28}$$

one can show that at

$$\beta_{\mathrm{NS}} = \frac{\alpha}{\alpha(1+2q)-1}\,, \tag{29}$$

we have $|\lambda_{1,2}(\beta_{\mathrm{NS}})| = 1$, such that $\lambda_{1,2} \neq 1$ and $\lambda_{1,2} \neq -1$. That is, $\lambda_{1,2} = \cos(\phi) \pm \mathrm{i}\sin(\phi), \phi \in (0,\pi)$ which indicates that a Neimark-Sacker bifurcation occurs. The crossing angle

$$\phi = \arccos\left(1 - \frac{\alpha^2(1+2q)}{2\big[\alpha(1+2q)-1\big]}\right), \tag{30}$$

can be obtained by substituting (29) into (26). Using (28) and (29) one can derive

$$\left.\frac{d|\lambda_1|}{d\beta}\right|_{\beta_{\mathrm{SN}}} = \frac{\alpha(1+2q)-1}{2} > 0\,. \tag{31}$$

Here the positive sign indicates that, as β is increased, the pair of complex conjugate eigenvalues moves outside the unit circle and the system loses stability. This can be observed in Fig. 1 for $d = 1, q = 1, \alpha = 1/2$ where (29) yields $\beta_{\mathrm{NS}} = 1$ as indicated by a cross.
Using the matrix

$$\mathbf{T}(\beta) = \big[\mathrm{Re}\,\mathbf{s}_1(\beta) \quad \mathrm{Im}\,\mathbf{s}_1(\beta) \quad \mathbf{s}_3\big]\,, \tag{32}$$

we can define the coordinate transformation

$$\begin{bmatrix}\Delta c\\ \Delta f\\ \Delta g\end{bmatrix} = \mathbf{T}(\beta)\begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix}\,. \tag{33}$$

Substituting (33) into (24) we obtain the form

$$\begin{bmatrix}\xi_{t+1}\\ \eta_{t+1}\\ \zeta_{t+1}\end{bmatrix} = \begin{bmatrix}\mathrm{Re}\lambda_1(\beta) & \mathrm{Im}\lambda_1(\beta) & 0\\ -\mathrm{Im}\lambda_1(\beta) & \mathrm{Re}\lambda_1(\beta) & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_t\\ \eta_t\\ \zeta_t\end{bmatrix} + \begin{bmatrix}a_{30}(\beta)\xi_t^3 + a_{21}(\beta)\xi_t^2\eta_t + a_{12}(\beta)\xi_t\eta_t^2 + a_{03}(\beta)\eta_t^3\\ b_{30}(\beta)\xi_t^3 + b_{21}(\beta)\xi_t^2\eta_t + b_{12}(\beta)\xi_t\eta_t^2 + b_{03}(\beta)\eta_t^3\\ \cdots\end{bmatrix}\,. \tag{34}$$

Again, since the first two equations only contain third-order terms, the center manifold reduction is not necessary and the third equation can be neglected. Applying the near-identity nonlinear transformation

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix} = \mathbf{I}\begin{bmatrix}u\\ v\end{bmatrix} + \begin{bmatrix}\sum_{i+j=3} s_{ij}(\beta)u^i v^j\\ \sum_{i+j=3} t_{ij}(\beta)u^i v^j\end{bmatrix}\,, \tag{35}$$

where $\mathbf{I}$ is the identity matrix, some third-order terms can be removed and we obtain the normal form

$$\begin{bmatrix}u_{t+1}\\ v_{t+1}\end{bmatrix} = r(\beta)\begin{bmatrix}\cos\psi(\beta) & \sin\psi(\beta)\\ -\sin\psi(\beta) & \cos\psi(\beta)\end{bmatrix}\begin{bmatrix}u_t\\ v_t\end{bmatrix} + (u_t^2+v_t^2)\begin{bmatrix}\cos\psi(\beta) & \sin\psi(\beta)\\ -\sin\psi(\beta) & \cos\psi(\beta)\end{bmatrix}\begin{bmatrix}\delta(\beta) & \kappa(\beta)\\ -\kappa(\beta) & \delta(\beta)\end{bmatrix}\begin{bmatrix}u_t\\ v_t\end{bmatrix}\,, \tag{36}$$

where $r(\beta_{\mathrm{NS}}) = 1$ and $\psi(\beta_{\mathrm{NS}}) = \phi$ (cf. (26,30)), while the coefficients $\delta(\beta), \kappa(\beta)$ are obtained from $a_{ij}(\beta), b_{ij}(\beta)$.
Close to the bifurcation point, that is, for $\beta \approx \beta_{\mathrm{NS}}$, the oscillations can be written as

$$\begin{bmatrix}u_t\\ v_t\end{bmatrix} = \sqrt{-\frac{\left.\frac{\mathrm{d}|\lambda_1|}{\mathrm{d}\beta}\right|_{\beta_{\mathrm{SN}}}}{\delta(\beta_{\mathrm{SN}})}(\beta-\beta_{\mathrm{SN}})}\cdot\begin{bmatrix}\cos\phi t\\ -\sin\phi t\end{bmatrix}\,. \tag{37}$$

Here the period of oscillations is given by $2\pi/\phi$. For example, for the parameters considered in Fig. 1 we have $2\pi/\phi \approx 4.77$ days. Notice the equivalency between (17,18) and (36,37) when $\phi = \pi$. In the former case we have $\mathrm{d}|\lambda_1|/\mathrm{d}\beta\big|_{\beta_{\mathrm{F}}} = -\mathrm{d}\lambda_1/\mathrm{d}\beta\big|_{\beta_{\mathrm{F}}}$.
That is, the criticality of the bifurcation is given by

$$\delta(\beta_{\mathrm{SN}}) = \frac{1}{8}\Big(\big(3a_{30}+a_{12}+b_{21}+3b_{03}\big)\cos\phi + \big(3a_{03}+a_{21}-b_{12}-3b_{30}\big)\sin\phi\Big) = -\frac{\alpha^2 q\big[2\alpha q(1+2q)-2q-3\big]}{d^2\big[\alpha(1+2q)-1\big]}\,, \tag{38}$$

where we used the abbreviated notation $\phi = \psi(\beta_{\mathrm{NS}})$, $a_{ij} = a_{ij}(\beta_{\mathrm{NS}})$, $b_{ij} = b_{ij}(\beta_{\mathrm{NS}})$. (A more general formula for cases including second-order terms can be found in the Appendix.) Again, we have

$$\begin{cases}\delta(\beta_{\mathrm{NS}}) > 0 \quad \text{if} \quad \alpha_0 < \alpha < \frac{3+2q}{2q(1+2q)} & \Rightarrow \text{ subcritical Neimark-Sacker}\,,\\ \delta(\beta_{\mathrm{NS}}) < 0 \quad \text{if} \quad \alpha > \frac{3+2q}{2q(1+2q)} & \Rightarrow \text{ supercritical Neimark-Sacker}\,.\end{cases} \tag{39}$$

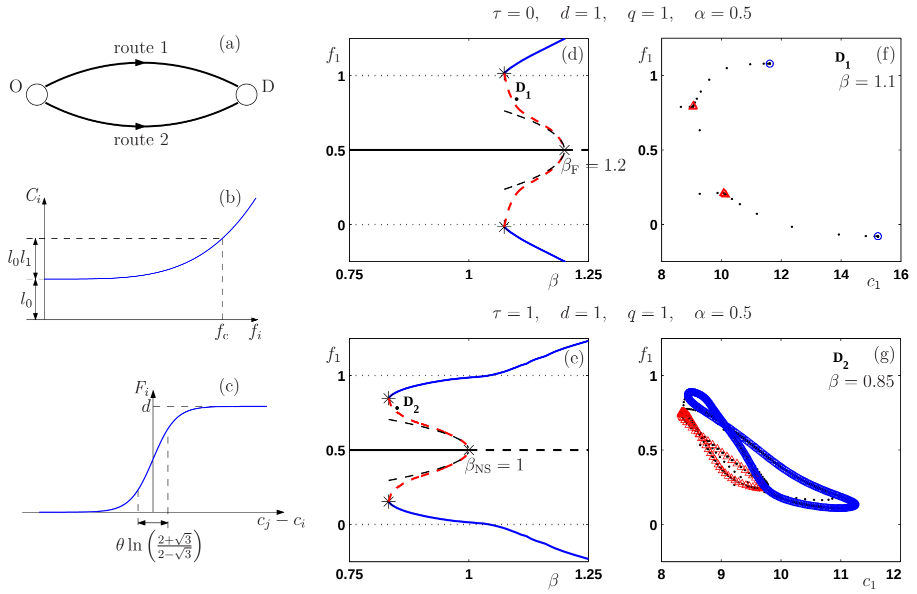

Figure 1: (a): Two-route traffic network with one origin-destination pair. (b): BPR cost function given by (2). (c): Logit-type network loading function given by (2). (d,e): Bifurcations diagrams for flip and Neimark-Sacker bifurcations showing the minima and maxima of oscillations together with the equilibrium (horizontal line at $d/2$). Solid and dashed curves represent stable and unstable states, respectively. Black curves correspond to analytical normal form calculations while colored curves are obtained by numerical continuation. Crosses represent flip and Neimark-Sacker bifurcations while fold bifurcations are denoted by stars. (f,g): Simulation results corresponding to the points D_1 and D_2. In the vicinity of stable and unstable periodic orbits the trajectories are highlighted by red triangles and blue circles, respectively.

Applying the transformations (33) and (35) at $\beta = \beta_{NS}$ and restricting the system to the center manifold $\zeta = 0$, we have $\Delta f \approx T_{21}(\beta_{NS})\xi + T_{22}(\beta_{NS})\eta \approx T_{21}(\beta_{NS})u + T_{22}(\beta_{NS})v$. Consequently, using (31,37,38) and noticing that $\big(T_{21}(\beta_{NS})\big)^2 + \big(T_{22}(\beta_{NS})\big)^2 = 1$ and $T_{21}(\beta_{NS}) = \cos\phi$ we obtain

$$\Delta f_t = \sqrt{S_{NS}(\beta - \beta_{NS})} \cdot \cos\big(\phi(t+1)\big) \quad \Rightarrow \quad f_{1,t} = d/2 + \sqrt{S_{NS}(\beta - \beta_{NS})} \cdot \cos\big(\phi(t+1)\big), \tag{40}$$

where

$$S_{NS} = \frac{d^2\big[\alpha(1+2q) - 1\big]^2}{2\alpha^2 q\big[2\alpha q(1+2q) - 2q - 3\big]}. \tag{41}$$

Note that since $T_{31}(\beta_{NS}) = 1$ and $T_{32}(\beta_{NS}) = 0$ we obtain the same amplitude for Δg but the phase shift is zero in this case which corresponds to the definition (23).

The corresponding minima and maxima of the oscillations are plotted in Fig. 1(e) for $d = 1, q = 1, \alpha = 1/2$. Since $\alpha < 5/6$, according to (39) the bifurcation is subcritical and indeed (38) yields $\delta(\beta_{NS}) = 1 > 0$ while (41) results in $S_{NS} = -1/4 < 0$.

Numerical verification

Due to subcriticality the arising oscillations are unstable in the vicinity of the stability losses as shown in Fig. 1(d,e) for the parameters indicated. Indeed, the analytical results given by (21,22,40,41) (black curves) match the numerical results (colored curves) in the vicinity of the bifurcations (crosses). The numerical results are obtained using numerical continuation namely the package MATCONT [4] for flip bifurcation and a modified version of the method [3] for Neimark-Sacker bifurcation. When deriving the numerical results we used $l_0 = 8, l_1 = 1, \theta = 1, f_c = 1$ to obtain $q = 1$. Notice that the subcriticality leads to bistability since the oscillatory branches fold back resulting in stable large-amplitude oscillations. Both the linearly unstable and bistable regimes are larger in the delayed case.

In the bistable regime the equilibrium is linearly stable but large perturbations may lead to oscillations as shown in Fig. 1(f,g) for the points marked D_1 and D_2. Here, the initial conditions are chosen such that the trajectories stay close to the unstable orbits (red triangles) before approaching the stable ones (blue circles). Notice that the state space behavior given by the flip bifurcation in the non-delayed case is fundamentally different from that given by the Neimark-Sacker bifurcation in the delayed case. Without delay the system oscillates between two states (after the transients decay) as shown in Fig. 1(f), while with delay quasi-periodic oscillation arise as shown by the cycles in Fig. 1(g).

Conclusion

A day-to-day assignment problem with driver experience delay was investigated in this paper that was formulated as a discrete-time nonlinear dynamical system. It was shown that the user equilibrium may lose its stability as the driver parameters are varied. For zero delays the stability loss occurs through flip bifurcation that leads to flow oscillations with period 2 days. However, having 1 day delay in the decision making results in oscillations of period that is much larger than the delay itself. Moreover, we observed bistable behavior where, depending on the initial conditions the system may approach the user equilibrium or exhibit large amplitude oscillations. Such behavior is more pronounced in the delayed case. We remark that expansion of bistable regimes and increase in oscillation periods can also be observed in car-following models when incorporating driver reaction time [6].

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Appendix – Normal forms for flip and Neimark-Sacker bifurcations

Here present how the nonlinear terms can be removed using nonlinear near identity transformation to obtain the normal forms for flip and Neimark-Sacker bifurcations when second-order terms are incorporated. We consider that the system has already restricted to the center manifold and, for simplicity, we consider the system at the critical point ($\beta = \beta_{\rm F}$ for flip and $\beta = \beta_{\rm NS}$ for Neimark-Sacker bifurcation). We use an abbreviated notation where we do not spell out the β dependence of the coefficients.

In case of flip bifurcation we start from the form

$$\xi_{t+1} = -\xi_t + a_2\xi_t^2 + a_3\xi_t^3\,, \tag{42}$$

cf. (16) at $\beta = \beta_{\rm F}$ where the second-order term is missing. Using the near-identity transformation

$$\xi = u + s_2 u^2\,, \tag{43}$$

one can show that by choosing the coefficient $s_2 = a_2/2$ the second-order term can be removed and we obtain the normal form

$$u_{t+1} = -u_t - \delta u_t^3\,, \tag{44}$$

where

$$\delta = -(a_3 + a_2^2)\,. \tag{45}$$

In case of Neimark-Sacker bifurcation we start with the form

$$\begin{bmatrix}\xi_{t+1}\\ \eta_{t+1}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\xi_t\\ \eta_t\end{bmatrix} + \begin{bmatrix}a_{20}\xi_t^2 + a_{11}\xi_t\eta_t + a_{02}\eta_t^2 + a_{30}\xi_t^3 + a_{21}\xi_t^2\eta_t + a_{12}\xi_t\eta_t^2 + a_{03}\eta_t^3\\ b_{20}\xi_t^2 + b_{11}\xi_t\eta_t + b_{02}\eta_t^2 + b_{30}\xi_t^3 + b_{21}\xi_t^2\eta_t + b_{12}\xi_t\eta_t^2 + b_{03}\eta_t^3\end{bmatrix}\,, \tag{46}$$

cf. (16) at $\beta = \beta_{\rm NS}$ where the second-order terms are missing. Applying the near-identity transformation

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix} = \mathbf{I}\begin{bmatrix}u\\ v\end{bmatrix} + \begin{bmatrix}\sum_{i+j=2,3} s_{ij}u^iv^j\\ \sum_{i+j=2,3} t_{ij}u^iv^j\end{bmatrix}\,, \tag{47}$$

we can remove all second-order terms and some of the third order terms by choosing the 14 coefficients $s_{ij}, t_{ij}, i+j = 2, 3$ appropriately. This leads to the normal from

$$\begin{bmatrix}u_{t+1}\\ v_{t+1}\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}u_t\\ v_t\end{bmatrix} + (u_t^2 + v_t^2)\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\delta & \kappa\\ -\kappa & \delta\end{bmatrix}\begin{bmatrix}u_t\\ v_t\end{bmatrix}\,, \tag{48}$$

where

$$\begin{aligned}\delta = {} & \frac{1}{8}\Big(\big(3a_{30} + a_{12} + b_{21} + 3b_{03}\big)\cos\phi + \big(3a_{03} + a_{21} - b_{12} - 3b_{30}\big)\sin\phi\Big)\\ & + \frac{1}{16}\bigg(- \big(2a_{20}a_{02} + 2b_{20}b_{02} + a_{11}^2 + b_{11}^2\big)\\ & \quad + 2\big(a_{20}^2 + b_{02}^2\big)\big(1 - 2\cos\phi\big)\cos\phi - 2\big(a_{02}^2 + b_{20}^2\big)\big(1 + \cos\phi\big)\big(3 - 2\cos\phi\big)\\ & \quad + \big(a_{20}b_{11} + a_{11}b_{02}\big)\big(5 + 2\cos\phi - 4\cos^2\phi\big) + \big(a_{11}b_{20} + a_{02}b_{11}\big)\big(1 + 2\cos\phi - 4\cos^2\phi\big)\\ & \quad - \Big(\big(a_{20} + a_{02}\big)\big(- a_{11} + b_{20} - b_{02}\big) + \big(b_{20} + b_{02}\big)\big(a_{20} - a_{02} + b_{11}\big)\Big)\frac{\big(1 - 6\cos\phi + 4\cos^2\phi\big)\sin\phi}{1 - \cos\phi}\bigg)\,.\end{aligned} \tag{49}$$