2016 IEEE 55th Conference on Decision and Control (CDC)
ARIA Resort & Casino
December 12-14, 2016, Las Vegas, USA

Stability of continuous-time systems with stochastic delay

Mehdi Sadeghpour and Gábor Orosz

***Abstract*— The stability of linear continuous-time systems with stochastic delay is investigated in this paper. The delay is assumed to be a piece-wise constant function of time such that it switches between finitely many different values stochastically. The stability of the stochastic system is assessed in terms of the convergence of the second moment of the state. Using infinite-dimensional solution operators, a stochastic linear map is constructed, allowing us to derive necessary and sufficient conditions of second moment stability. The discretization of the solution operators can be used to draw stability charts. An illustrative example is discussed to shed some light on the effects of stochastic delays on stability.**

I. INTRODUCTION

The problem of the stability analysis and control design for systems with stochastic delays has been raised in many applications. For instance, in networked control systems, data are often transferred with random communication delay [1], [2]. In connected vehicle systems, the information from vehicles ahead is received at random times due to packet loss in wireless communication. Also, driver reaction time may change stochastically with time [3], [4]. Random delays also arise in gene regulatory networks since the execution times of transcription and translation processes are influenced by the noisy cell environment [5]–[7].

The stability analysis of discrete-time systems with stochastic delay has been investigated thoroughly in literature, e.g., [1], [2], [8], [9]. The stability analysis of linear and nonlinear continuous-time systems with stochastic delay has been studied using Lyapunov-based theorems [10]–[14]. Such theorems result in sufficient stability conditions that are typically very conservative. To avoid conservatism, we present necessary and sufficient stability conditions for continuous-time linear systems with stochastic delay. In [15] the authors derived necessary and sufficient conditions for some very specific stochastic delay variations. Indeed, the problem of the stability analysis of a continuous-time system with stochastic delay may be essentially different depending on the form of the delay variations as well as the definition of a "stable" system.

In this paper, we provide necessary and sufficient conditions for the second moment stability of linear continuous-time systems with stochastic delay. The delay trajectories are assumed to be piece-wise constant functions of time. The delay can assume finitely many values and switches between these values based on a fixed probability distribution. We derive the time evolution of the second moment of the state using infinite-dimensional solution operators. By calculating the spectral radii of these operators we construct necessary and sufficient conditions for stability. By discretizing the solution operators the stability conditions can be evaluated numerically allowing us to construct stability charts. The tools developed are demonstrated with a case study.

Mehdi Sadeghpour and Gábor Orosz are with the Department of Mechanical Engineering at the University of Michigan, Ann Arbor, MI, 48109, USA mehsad@umich.edu, orosz@umich.edu

II. SOLUTION OPERATORS FOR SYSTEMS WITH STOCHASTIC DELAYS

We consider a linear system of the form

$$\dot{x}(t) = \mathbf{a}x(t) + \mathbf{b}x(t-\tau(t)), \tag{1}$$

where the dot stands for differentiation with respect to time t, $x \in \mathbb{R}^n$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{n\times n}$, and the delay $\tau(t)$ is a stochastic process. We assume that $\tau(t) \in \{\tau_1, \tau_2, \ldots, \tau_J\}$, $\forall\, t \geq 0$ where $0 < \tau_1 < \tau_2 < \ldots < \tau_J$. The initial condition is given by

$$x(\theta) = \phi(\theta), \quad -\tau_J \leq \theta \leq 0, \tag{2}$$

where $\phi \in \mathcal{C}\big([-\tau_J, 0], \mathbb{R}^n\big)$.

We assume that the trajectories of the delay are such that the delay remains constant in the time intervals $[kT, (k+1)T)$, $k = 0, 1, 2, \ldots$, and switches to new values (possibly the same value) at times $t = kT$. We call T the delay holding time. The switches occur based on the probability distribution $w = [w_1\; w_2\; \ldots\; w_J]$ where w_j is the probability of switching to the delay τ_j. Also, we assume that $T < \tau_1$. This assumption implies that the delay switching rate is relatively fast (with respect to the delay values). In Fig. 1, two sample trajectories of the delay $\tau(t)$ as well as the corresponding trajectories of the state $x(t)$ are exhibited. Since the probability distribution vector w is fixed at all switches, the switches are independent, identically distributed (i.i.d.). The goal is to study the stability of the trivial solution of (1,2). Throughout this paper, we study the convergence of the second moment $\mathbb{E}[x(t)x^{\mathrm{T}}(t)]$ to zero (as $t \to \infty$) as a measure of stability, where $\mathbb{E}$ denotes the expected value and T denotes transpose. We refer to it as second moment stability (also known as mean-square stability). We will also give conditions for the convergence of the mean $\mathbb{E}[x(t)]$ to zero along the way.

We start with constructing the solution operators for system (1,2). We recall that for the deterministic delay differential equation

$$\begin{aligned} &\dot{x}(t) = \mathbf{a}x(t) + \mathbf{b}x(t-\tau), \\ &x(\theta) = \phi(\theta), \quad -\tau \leq \theta \leq 0, \end{aligned} \tag{3}$$

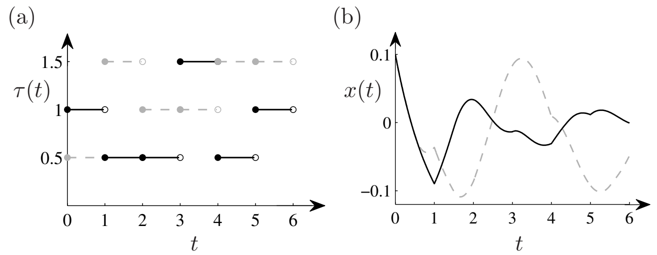

Fig. 1. (a) Two sample paths of the delay with holding time $T = 1$ and $J = 3$ possible delay values. (b) Trajectories of system (1) with $a = -1$ and $b = -2$ corresponding to sample paths of the delay shown in Fig. 1(a) and initial condition $\phi(\theta) = 0.1$, $-1.5 \le \theta \le 0$.

where $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, the solution operator is defined by

$$(\mathcal{T}(t)\phi)(\theta) = x_t(\theta) = x(t+\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0. \tag{4}$$

The operator $\mathcal{T}(t) : \mathcal{C}([-\tau, 0], \mathbb{R}^n) \to \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, $\forall\, t \ge 0$, is a bounded, linear operator and the family of operators $\mathcal{T}(t)$, $t \ge 0$, is a strongly continuous semigroup [16].

Now consider the following J deterministic systems

$$\begin{aligned} &\dot{x}(t) = \mathbf{a}x(t) + \mathbf{b}x(t - \tau_j), \\ &j = 1, \ldots, J \end{aligned} \tag{5}$$

with their respective solution operators $\mathcal{T}_j(t) : \mathcal{C}([-\tau_j, 0], \mathbb{R}^n) \to \mathcal{C}([-\tau_j, 0], \mathbb{R}^n)$, $\forall\, t \ge 0$. Note that these operators have different domains due to different delay values. As mentioned above, in the stochastic system (1), along each interval $[kT, (k+1)T)$, $k = 0, 1, 2, \ldots$, the delay is kept fixed; therefore, one of the J deterministic systems in (5) governs the evolution of system (1). Hence we consider the operators $\mathcal{T}_j(T)$ that progress the solution T units of time forward, and extend their definitions such that they all have a common domain. For any $\phi \in \mathcal{C}([-\tau_J, 0], \mathbb{R}^n)$, we define

$$\begin{aligned} &\phi_j(\theta) := \phi(\theta), \quad -\tau_j \le \theta \le 0, \\ &\phi_j \in \mathcal{C}([-\tau_j, 0], \mathbb{R}^n), \quad j = 1, \ldots, J-1, \end{aligned} \tag{6}$$

and construct the following operators

$$\begin{aligned} &(\mathcal{Q}_j\phi)(\theta) = \begin{cases} \big(\mathcal{T}_j(T)\phi_j\big)(\theta) & -T \le \theta \le 0 \\ \phi(\theta + T) & -\tau_J \le \theta < -T \end{cases} \\ &j = 1, \ldots, J-1, \end{aligned} \tag{7}$$

and

$$(\mathcal{Q}_J\phi)(\theta) = \big(\mathcal{T}_J(T)\phi\big)(\theta), \quad -\tau_J \le \theta \le 0, \tag{8}$$

where $\mathcal{Q}_j : \mathcal{C}([-\tau_J, 0], \mathbb{R}^n) \to \mathcal{C}([-\tau_J, 0], \mathbb{R}^n)$, $\forall\, j \in \{1, \ldots, J\}$. Note that the new operators $\mathcal{Q}_j$ are defined by augmenting the operators $\mathcal{T}_j(T)$ with a shift. The operators $\mathcal{Q}_j$ can be used to describe the dynamics of (1).

From now on, we denote the space $\mathcal{C}([-\tau_J, 0], \mathbb{R}^n)$ by $\mathcal{C}$ in order to simplify the notation. Now suppose that the stochastic system (1) is realized up to the time $t = kT$ and define the "state" of the system at $t = kT$ as

$$x_{kT}(\theta) = x(kT + \theta), \quad -\tau_J \le \theta \le 0, \tag{9}$$

where $x_{kT} \in \mathcal{C}$. Let us assume $\tau(t) = \tau_j$ in the time interval $\big[kT, (k+1)T\big)$ where τ_j is one of the values in the set $\{\tau_1, \ldots, \tau_J\}$. Using the solution operators (7)-(8), we can express the "state" at $t = (k+1)T$ by

$$x_{(k+1)T} = \mathcal{Q}_j x_{kT}. \tag{10}$$

Since the probability that $\tau(t) = \tau_j$ in the time interval $\big[kT, (k+1)T\big)$ is w_j, $j \in \{1, 2, \ldots, J\}$, one can form the following stochastic map

$$x_{(k+1)T} = \mathcal{Q}_k x_{kT}, \tag{11}$$

where $\mathcal{Q}_k : \mathcal{C} \to \mathcal{C}$, $x_{kT} \in \mathcal{C}$, $\forall\, k \in \{0, 1, 2, \ldots\}$, and $\mathcal{Q}_k = \mathcal{Q}_j$ with probability w_j, $\forall\, j \in \{1, \ldots, J\}$, and $\forall\, k \in \{0, 1, 2, \ldots\}$. In the next section, we study the stability of the trivial solution of (11), and we will show that the stability of (11) and (1) are equivalent.

III. STABILITY ANALYSIS

In this section, we analyze the stability of (11). We first give a definition of exponential (or power) stability that is used in this paper. Then we provide two theorems, one for the stability of the mean and one for the stability of the second moment of (11). Finally, we show that the stability of (11) and (1) are equivalent.

Definition 1. System

$$x_{k+1} = \mathcal{Q}x_k, \tag{12}$$

with $\mathcal{Q} : \mathbb{B} \to \mathbb{B}$, ($\mathbb{B}$ is a Banach space) and initial condition $x_0 \in \mathbb{B}$, where $\mathcal{Q}$ is a bounded, linear operator, is exponentially (or power) stable if for every $x_0 \in \mathbb{B}$ there exist $M \ge 1$ and $0 \le r < 1$ such that

$$\|x_k\| \le M r^k, \quad k = 0, 1, 2, \ldots \tag{13}$$

holds; see [17].

The following lemma provides a classical result regarding the relationship between exponential stability and the spectral radius of operators (for example see Theorem 2.1 in [17]).

Lemma 1. Let $\sigma(\mathcal{Q})$ denote the spectrum of $\mathcal{Q}$ and $\rho(\mathcal{Q})$ the spectral radius of $\mathcal{Q}$, i.e.,

$$\rho(\mathcal{Q}) = \sup\big\{|\lambda| : \lambda \in \sigma(\mathcal{Q})\big\}. \tag{14}$$

Then system (12) is exponentially stable if and only if

$$\rho(\mathcal{Q}) < 1. \tag{15}$$

Using Lemma 1, we can find the conditions of the stability of the mean of system (11). Taking the expected value of both sides of (11) we have

$$\mathbb{E}[x_{(k+1)T}] = \mathbb{E}[\mathcal{Q}_k x_{kT}]. \tag{16}$$

Due to the i.i.d. property of the delays, the delay in the time interval $\big[kT, (k+1)T\big)$ is independent of the delay and the state in the time interval $\big[(k-1)T, kT\big)$. Therefore, the operator $\mathcal{Q}_k$, which only depends on the delay value in the time interval $\big[kT,(k+1)T\big)$, is independent of x_{kT}. Thus, from (16) we arrive at

$$\mathbb{E}[x_{(k+1)T}] = \mathbb{E}[\mathcal{Q}_k]\mathbb{E}[x_{kT}]. \tag{17}$$

where $\mathbb{E}[\mathcal{Q}_k] : \mathcal{C} \to \mathcal{C}$ is

$$\mathbb{E}[\mathcal{Q}_k] = \sum_{j=1}^{J} w_j \mathcal{Q}_j. \tag{18}$$

The following theorem states a necessary and sufficient condition for the stability of the mean of system (11).

Theorem 1. Consider system (11) where the initial condition is given as

$$x_0(\theta) = \phi(\theta), \quad -\tau_J \le \theta \le 0, \quad \phi \in \mathcal{C}, \tag{19}$$

and the probability that $\mathcal{Q}_k = \mathcal{Q}_j$ is w_j, $\forall\, j \in \{1,\dots,J\}$, $\forall\, k \in \{0,1,2,\dots\}$. Then there exist $M \ge 1$ and $0 \le r < 1$ such that

$$\big\|\mathbb{E}[x_{kT}]\big\|_{\sup} \le M r^k,\ \forall\, k \in \{0,1,2,\dots\} \tag{20}$$

if and only if

$$\rho\Big(\sum_{j=1}^{J} w_j \mathcal{Q}_j\Big) < 1. \tag{21}$$

Here the $\mathcal{Q}_j$'s are given by (7)-(8) and $\|.\|_{\sup}$ denotes the sup norm on $\mathcal{C}$, i.e.,

$$\|\phi\|_{\sup} = \sup_{-\tau_J \le \theta \le 0} \|\phi(\theta)\|_{\max}, \quad \phi \in \mathcal{C}, \tag{22}$$

where $\|.\|_{\max}$ denotes the max norm on $\mathbb{R}^n$, i.e.,

$$\|x\|_{\max} = \max_{i=1,\dots,n} |x^i|, \tag{23}$$

where $x \in \mathbb{R}^n$ and x^i is the i^{th} element of x.

Proof. The proof is immediately obtained by the application of Lemma 1 on (17). ☐

Next, we want to analyze the stability of the second moment of (11). To describe the dynamics of the second moment, we form the tensor product of the map (11) with itself. We use the injective tensor product since the injective norm proves useful in analyzing the second moment. Let $\mathcal{C}\hat{\otimes}_\epsilon\mathcal{C}$ denote the completion of the tensor product of the vector space $\mathcal{C}$ with itself under the injective norm defined as

$$\|u\|_{\text{inj}} = \sup\Big\{\Big|\sum_{m=1}^{M} f(\phi_m) g(\psi_m)\Big| :\ f, g \in B_{\mathcal{C}^*},\Big\} \tag{24}$$

where $u = \sum_{m=1}^{M} \phi_m \otimes \psi_m$ is a tensor in $\mathcal{C}\hat{\otimes}_\epsilon\mathcal{C}$, and $B_{\mathcal{C}^*}$ is the closed unit ball on $\mathcal{C}^*$ (the normed dual of $\mathcal{C}$). In other words, f and g are bounded, linear functionals defined on $\mathcal{C}$ with norm less than or equal to 1, and the supremum in (24) is taken over all such f and g; see [18].

An important observation in the definition of the injective norm in (24) is that one can substitute $B_{\mathcal{C}^*}$ with a norming set [18]. A subset $\mathcal{N}$ of $B_{\mathcal{C}^*}$ is said to be a norming set if $\|\phi\|_s = \sup\{|f(\phi)| : f \in \mathcal{N}\}$ for every $\phi \in \mathcal{C}$. We will use this observation to manipulate the definition of the injective norm in a way that is more useful toward our purposes in analyzing second moment dynamics. Let us define the following norm on $\mathcal{C}\hat{\otimes}_\epsilon\mathcal{C}$

Definition 2. Let $u = \sum_{m=1}^{M} \phi_m \otimes \psi_m$ belong to $\mathcal{C}\hat{\otimes}_\epsilon\mathcal{C}$. Then we define the norm

$$\|u\|_\epsilon = \sup_{\substack{-\tau_J \le \theta_1,\theta_2 \le 0 \\ i_1,i_2=1,\dots,n}} \Big|\sum_{m=1}^{M} \phi_m^{i_1}(\theta_1)\, \psi_m^{i_2}(\theta_2)\Big|. \tag{25}$$

The following lemma shows that $\|.\|_\epsilon$ is the injective norm on $\mathcal{C}\hat{\otimes}_\epsilon\mathcal{C}$.

Lemma 2. The norm $\|.\|_\epsilon$, defined in (25), is equivalent to the injective norm $\|.\|_{\text{inj}}$, defined in (24).

Proof. Let $\phi \in \mathcal{C}$. Consider the linear functionals

$$\delta_\theta^i(\phi) = \phi^i(\theta), \tag{26}$$

where $\phi^i(\theta)$ is the i^{th} component of $\phi(\theta)$, $i = 1,\dots,n$ and $-\tau_J \le \theta \le 0$. The set $\mathcal{N} = \big\{\delta_\theta^i : -\tau_J \le \theta \le 0,\, i = 1,\dots,n\big\}$ is a subset of $B_{\mathcal{C}^*}$. Notice that we have $|\delta_\theta^i(\phi)| = |\phi^i(\theta)| \le \|\phi(\theta)\|_{\max} \le \|\phi\|_{\sup}$, $\forall\, \phi \in \mathcal{C}$. Then, using the definition of the norm of a functional $|\delta_\theta^i(\phi)| \le \|\delta_\theta^i\|\|\phi\|_{\sup}$, we obtain $\|\delta_\theta^i\| \le 1$. The set $\mathcal{N}$ is also a norming set, because for every $\phi \in \mathcal{C}$

$$\begin{aligned}\|\phi\|_{\sup} &= \sup_{-\tau_J \le \theta \le 0} \|\phi(\theta)\|_{\max} = \sup_{-\tau_J \le \theta \le 0} \big\{\max_{i=1,\dots,n} |\phi^i(\theta)|\big\}\\ &= \sup\Big\{\big|\delta_\theta^i(\phi)\big| : \delta_\theta^i \in \mathcal{N}\Big\}.\end{aligned} \tag{27}$$

Therefore, we can substitute $B_{\mathcal{C}^*}$ with $\mathcal{N}$ in (24) that yields (25). ☐

Note that there are J deterministic maps associated with system (11) if $\mathcal{Q}_k = \mathcal{Q}_j$, $j \in \{1,\dots,J\}$. Corresponding to each deterministic map, there exists a unique bounded, linear operator $\mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j : \mathcal{C}\hat{\otimes}_\epsilon\mathcal{C} \to \mathcal{C}\hat{\otimes}_\epsilon\mathcal{C}$ such that $(\mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j)(\phi \otimes \psi) = (\mathcal{Q}_j\phi) \otimes (\mathcal{Q}_j\psi)$ for every $\phi, \psi \in \mathcal{C}$ (see Proposition 3.2 in [18]). Therefore, if $\tau(t) = \tau_j$ in the time interval $\big[kT,(k+1)T\big)$, which means $\mathcal{Q}_k = \mathcal{Q}_j$, then we have

$$\begin{aligned}(\mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j)(x_{kT} \otimes x_{kT}) &= (\mathcal{Q}_j x_{kT}) \otimes (\mathcal{Q}_j x_{kT})\\ &= x_{(k+1)T} \otimes x_{(k+1)T}.\end{aligned} \tag{28}$$

Now, using (28), we can form the stochastic map

$$x_{(k+1)T} \otimes x_{(k+1)T} = (\mathcal{Q}_k \otimes_\epsilon \mathcal{Q}_k)(x_{kT} \otimes x_{kT}), \tag{29}$$

where $\mathcal{Q}_k \otimes_\epsilon \mathcal{Q}_k = \mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j$ with probability w_j, $\forall\, j \in \{1,\dots,J\}$ and $\forall\, k \in \{0,1,2,\dots\}$.

Taking the expected value of both sides of (29) we arrive at

$$\mathbb{E}[x_{(k+1)T} \otimes x_{(k+1)T}] = \mathbb{E}\big[(\mathcal{Q}_k \otimes_\epsilon \mathcal{Q}_k)(x_{kT} \otimes x_{kT})\big]. \tag{30}$$

Due to independence of $\mathcal{Q}_k$ and x_{kT}, $\mathcal{Q}_k \otimes_\epsilon \mathcal{Q}_k$ is independent of $x_{kT} \otimes x_{kT}$ and we obtain

$$\mathbb{E}[x_{(k+1)T} \otimes x_{(k+1)T}] = \mathbb{E}\big[(\mathcal{Q}_k \otimes_\epsilon \mathcal{Q}_k)\big]\mathbb{E}\big[(x_{kT} \otimes x_{kT})\big], \tag{31}$$

where

$$\mathbb{E}\big[(\mathcal{Q}_k \otimes_\epsilon \mathcal{Q}_k)\big] = \sum_{j=1}^{J} w_j\, \mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j, \tag{32}$$

and $\mathbb{E}\big[(\mathcal{Q}_k \otimes_\epsilon \mathcal{Q}_k)\big] : \mathcal{C}\hat{\otimes}_\epsilon\mathcal{C} \to \mathcal{C}\hat{\otimes}_\epsilon\mathcal{C}$ is a bounded, linear operator. Now we are ready to state a theorem that provides a necessary and sufficient condition for the exponential stability of the second moment of (11).

Theorem 2. Consider system (11) where the initial condition is given as

$$x_0(\theta) = \phi(\theta), \quad -\tau_J \le \theta \le 0, \quad \phi \in \mathcal{C}, \tag{33}$$

and the probability of $\mathcal{Q}_k = \mathcal{Q}_j$ is w_j, $\forall\, j \in \{1, \ldots, J\}$ and $\forall\, k \in \{0, 1, 2, \ldots\}$. Then there exist $M \ge 1$ and $0 \le r < 1$ such that

$$\sup_{\substack{-\tau_J \le \theta_1, \theta_2 \le 0\\ i_1, i_2 = 1, \ldots, n}} \Big|\mathbb{E}\big[x_{kT}^{i_1}(\theta_1) x_{kT}^{i_2}(\theta_2)\big]\Big| \le M r^k, \tag{34}$$

$\forall\, k \in \{0, 1, 2, \ldots\}$ if and only if

$$\rho\Big(\sum_{j=1}^{J} w_j\, \mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j\Big) < 1, \tag{35}$$

where $\mathcal{Q}_j$'s are given by (7)-(8).

Proof. Applying Lemma 1 to (31), we can state that there exist $M \ge 1$ and $0 \le r < 1$ such that

$$\Big\|\mathbb{E}[x_{kT} \otimes x_{kT}]\Big\|_\epsilon \le M r^k, \tag{36}$$

$\forall\, k \in \{0, 1, 2, \ldots\}$ if and only if

$$\rho\Big(\sum_{j=1}^{J} w_j\, \mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j\Big) < 1. \tag{37}$$

From (25), we have

$$\Big\|\mathbb{E}\big[x_{kT} \otimes x_{kT}\big]\Big\|_\epsilon = \sup_{\substack{-\tau_J \le \theta_1, \theta_2 \le 0\\ i_1, i_2 = 1, \ldots, n}} \Big|\mathbb{E}\big[x_{kT}^{i_1}(\theta_1) x_{kT}^{i_2}(\theta_2)\big]\Big| \tag{38}$$

that completes the proof. □

The condition (35) is a necessary and sufficient condition for the second moment stability of system (11). In the following, we show that the second moment stability of the trivial solution of (11) is indeed equivalent to the second moment stability of system (1).

Corollary 1. Consider system (1)-(2) with the delay behavior as described in Section II. There exists $M \ge 1$ and $\omega > 0$ such that

$$\sup_{i_1, i_2 = 1, \ldots, n} \Big|\mathbb{E}\big[x^{i_1}(t) x^{i_2}(t)\big]\Big| \le M \mathrm{e}^{-\omega t}, \quad \forall\, t \ge 0, \tag{39}$$

if and only if

$$\rho\Big(\sum_{j=1}^{J} w_j\, \mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j\Big) < 1, \tag{40}$$

where $\mathcal{Q}_j$'s are given by (7)-(8).

Proof. Assume $\rho\Big(\sum_{j=1}^{J} w_j\, \mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j\Big) < 1$. Then from Theorem 2, there exists $M \ge 1$ and $0 \le r < 1$ such that (34) holds. For any $t \ge 0$, there exists $k \in \{1, 2, \ldots\}$, such that $(k-1)T \le t < kT$. Thus, by choosing $\theta_1 = \theta_2 = -kT + t$ in (34) and noting that $x_{kT}(\theta) = x(kT + \theta)$, we have

$$\sup_{i_1, i_2 = 1, \ldots, n} \Big|\mathbb{E}\big[x^{i_1}(t) x^{i_2}(t)\big]\Big| \le M r^k. \tag{41}$$

Also, since $t/T < k$, then $r^k < r^{t/T} = \mathrm{e}^{(\frac{1}{T}\log r)t}$. Thus, (41) can be written as

$$\sup_{i_1, i_2 = 1, \ldots, n} \Big|\mathbb{E}\big[x^{i_1}(t) x^{i_2}(t)\big]\Big| \le M \mathrm{e}^{-\tilde{\omega} t}, \quad \forall\, t \ge 0, \tag{42}$$

where $\tilde{\omega} = -\frac{1}{T}\log r$.

To show the reverse, assume that there exists $M \ge 1$ and $\omega > 0$ such that (39) holds. For any $k \in \{1, 2, \ldots\}$, consider the interval $[kT - \tau_J, kT]$. From (39), we have

$$\Big|\mathbb{E}\big[x_{kT}^{i}(-kT + t) x_{kT}^{i}(-kT + t)\big]\Big| \le M \mathrm{e}^{-\omega t}, \tag{43}$$

$\forall\, t \in [kT - \tau_J, kT]$ and $\forall\, i \in \{1, \ldots, n\}$. Let $\theta = -kT + t$ and observe that since $t \ge kT - \tau_J$, then $\mathrm{e}^{-\omega t} \le \mathrm{e}^{-\omega(kT - \tau_J)} = \mathrm{e}^{\omega\tau_J}(\mathrm{e}^{-\omega T})^k$. Therefore, (43) can be written as

$$\Big|\mathbb{E}\Big[\big(x_{kT}^{i}(\theta)\big)^2\Big]\Big| \le \tilde{M}\tilde{r}^k, \tag{44}$$

$\forall\, \theta \in [-\tau_J, 0]$ and $\forall\, i \in \{1, \ldots, n\}$, where $\tilde{M} = M\mathrm{e}^{\omega\tau_J}$ and $\tilde{r} = \mathrm{e}^{-\omega T}$. Now using Cauchy-Schwarz inequality, we have

$$\Big|\mathbb{E}\big[x_{kT}^{i_1}(\theta_1) x_{kT}^{i_2}(\theta_2)\big]\Big| \le \Big(\mathbb{E}\Big[\big(x_{kT}^{i_1}(\theta_1)\big)^2\Big]\mathbb{E}\Big[\big(x_{kT}^{i_2}(\theta_2)\big)^2\Big]\Big)^{\frac{1}{2}} \tag{45}$$

$\forall\, \theta_\ell \in [-\tau_J, 0]$ and $\forall\, i_\ell \in \{1, \ldots, n\}$, $\ell = 1, 2$. Using (44) in (45) we have

$$\Big|\mathbb{E}\big[x_{kT}^{i_1}(\theta_1) x_{kT}^{i_2}(\theta_2)\big]\Big| \le \tilde{M}\tilde{r}^k, \tag{46}$$

$\forall\, \theta_\ell \in [-\tau_J, 0]$ and $\forall\, i_\ell \in \{1, \ldots, n\}$, $\ell = 1, 2$, which implies

$$\sup_{\substack{-\tau_J \le \theta_1, \theta_2 \le 0\\ i_1, i_2 = 1, \ldots, n}} \Big|\mathbb{E}\big[x_{kT}^{i_1}(\theta_1) x_{kT}^{i_2}(\theta_2)\big]\Big| \le \tilde{M}\tilde{r}^k, \tag{47}$$

$\forall\, k \in \{0, 1, 2, \ldots\}$. Using Theorem 2, (47) implies

$$\rho\Big(\sum_{j=1}^{J} w_j\, \mathcal{Q}_j \otimes_\epsilon \mathcal{Q}_j\Big) < 1. \tag{48}$$

□

Remark 1. In our formulation in Section II, we assumed that the initial condition of system (1) is given. All results of the paper are also applicable in the case when the initial condition (the function $\phi \in \mathcal{C}$) is stochastic, as long as $\|\mathbb{E}[\phi]\|_{\sup}$ and $\sup_{\substack{-\tau_J \le \theta_1, \theta_2 \le 0\\ i_1, i_2 = 1, \ldots, n}} \Big|\mathbb{E}\big[\phi^{i_1}(\theta_1)\phi^{i_2}(\theta_2)\big]\Big|$ exist.

In practice, condition (21) of Theorem 1 and condition (35) of Theorem 2 cannot be investigated directly due to the infinite-dimensional nature of the relevant operators. Therefore, in the next section, we provide finite-dimensional versions of conditions (21) and (35).

IV. FINITE-DIMENSIONAL APPROXIMATIONS

There are a number of time-discretization techniques that can be used to obtain a finite-dimensional representation of the solution operator of a delay differential equation, such as the techniques proposed in [19] and [20]. Using such techniques, one can obtain finite-dimensional matrices $\mathbf{Q}_j$ as approximations of the infinite-dimensional operators $\mathcal{Q}_j$ defined by (7)-(8). Assume here that, after using a time-discretization technique, we obtain the finite-dimensional map

$$X_{k+1} = \mathbf{Q}_k X_k, \tag{49}$$

as an approximation of the infinite-dimensional map (11), where $\mathbf{Q}_k \in \mathbb{R}^{N\times N}$ and $X_k \in \mathbb{R}^N$ for some $N \in \mathbb{N}$ that is a parameter of the discretization scheme. Similar to (11), we have $\mathbf{Q}_k = \mathbf{Q}_j$ with probability w_j, $\forall\, j \in \{1,\ldots,J\}$ and $\forall\, k \in \{0,1,2,\ldots\}$.

Taking the expected value of both sides of (49) and using independence of $\mathbf{Q}_k$ and X_k, we have

$$\mathbb{E}[X_{k+1}] = \mathbb{E}[\mathbf{Q}_k]\mathbb{E}[X_k], \tag{50}$$

where

$$\mathbb{E}[\mathbf{Q}_k] = \sum_{j=1}^{J} w_j \mathbf{Q}_j, \tag{51}$$

which is a finite-dimensional version of (18).

Now we take the Kronecker product of each side of (49) with itself since the Kronecker product is the finite-dimensional version of the tensor product in infinite-dimensional spaces. We have

$$\begin{aligned} X_{k+1} \otimes X_{k+1} &= \mathbf{Q}_k X_k \otimes \mathbf{Q}_k X_k \\ &= (\mathbf{Q}_k \otimes \mathbf{Q}_k)(X_k \otimes X_k) \end{aligned} \tag{52}$$

where $X_k \otimes X_k \in \mathbb{R}^{N^2}$, $\mathbf{Q}_k \otimes \mathbf{Q}_k \in \mathbb{R}^{N^2\times N^2}$, and $\mathbf{Q}_k \otimes \mathbf{Q}_k = \mathbf{Q}_j \otimes \mathbf{Q}_j$ with probability w_j, $\forall\, j \in \{1,\ldots,J\}$ and $\forall\, k \in \{0,1,2,\ldots\}$. Taking the expected value of both sides of (52) and using the independence of $\mathbf{Q}_k$ and X_k, we arrive at

$$\mathbb{E}[X_{k+1} \otimes X_{k+1}] = \mathbb{E}[(\mathbf{Q}_k \otimes \mathbf{Q}_k)]\mathbb{E}[(X_k \otimes X_k)], \tag{53}$$

where

$$\mathbb{E}[(\mathbf{Q}_k \otimes \mathbf{Q}_k)] = \sum_{j=1}^{J} w_j \mathbf{Q}_j \otimes \mathbf{Q}_j, \tag{54}$$

which is a finite-dimensional version of (32).

From (50), we know that $\mathbb{E}[X_k]$ exponentially converges to zero if and only if

$$\rho(\mathbb{E}[\mathbf{Q}_k]) < 1. \tag{55}$$

Therefore, to investigate the stability of the mean of system (1), one can use (55) in lieu of (21) as an approximation.

Similarly, from (52), we know that $\mathbb{E}[X_k \otimes X_k]$ exponentially converges to zero if and only if

$$\rho(\mathbb{E}[\mathbf{Q}_k \otimes \mathbf{Q}_k]) < 1. \tag{56}$$

Thus, to investigate the stability of the second moment of (1), one can use (56) in lieu of (35) as an approximation. Note that in the finite dimensional setting, the second moment is usually represented with the matrix $\mathbb{E}[XX^{\mathrm{T}}]$ rather than $\mathbb{E}[X \otimes X]$ that we used above. However, the elements of $\mathbb{E}[XX^{\mathrm{T}}]$ are in a one-to-one correspondence with the elements of the vector $\mathbb{E}[X \otimes X]$ as we have $\mathbb{E}[X \otimes X] = vec(\mathbb{E}[XX^{\mathrm{T}}])$ where vec is an isometric isomorphism that puts the columns of a matrix below one another.

We know that the leading eigenvalues of the matrix $\mathbf{Q}_j$ approximate the leading eigenvalues of the operator $\mathcal{Q}_j$ and by decreasing the time step of the time-discretization technique, the approximation error can be made arbitrarily small (see for example [20] for details about the convergence of discretization techniques). Although, by making the time step small, the size of matrices $\mathbf{Q}_j \otimes \mathbf{Q}_j$ grow large quickly putting a limit on the smallest value that can be sued for the time step due to hardware limitations. The speed of the convergence of the eigenvalues depends on the parameters of the discretization method used. Conditions (55) and (56) are finite-dimensional approximations of conditions (21) and (35), respectively, as (50) and (53) are finite-dimensional versions of (17) and (31). Thus, the stability regions obtained based on conditions (55) and (56) will converge to those based on conditions (21) and (35) while the convergence properties follow those of the discretization method used. In practice, we gradually decrease the time step to observe the convergence of the stability charts.

V. AN EXAMPLE

In this section, we provide an example to illustrate the effects of stochastic delay on the stability. Consider a second order linear system with delayed feedback

$$\dot{x}(t) = \mathbf{a}x(t) + \mathbf{b}_1 u(t-\tau(t)), \tag{57}$$

where $x = [x_1\ \ x_2]^{\mathrm{T}} \in \mathbb{R}^{2\times 1}$, $u \in \mathbb{R}$, and

$$\mathbf{a} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{58}$$

The control law is

$$u = \begin{bmatrix} -k_1 & -k_2 \end{bmatrix} x, \tag{59}$$

with k_1 and k_2 as control gains. This example can for instance represent the position control of a mass-spring-damper system with delays in the feedback channel. Note that the pair $\mathbf{a}, \mathbf{b}_1$ is in the canonical reachable form and that the closed-loop system takes the form of (1) with

$$\mathbf{b} = \begin{bmatrix} 0 & 0 \\ -k_1 & -k_2 \end{bmatrix}. \tag{60}$$

Assume $\tau(t)$ is chosen from the set $\{0.7, 0.85, 1, 1.15, 1.3\}$ according to a uniform probability distribution $w_j = \frac{1}{5}, \forall j = 1,\ldots,5$ and the holding time is $T = 0.5$.

In this example, we use the semi-discretization technique that is proposed in [19] (see Section 3.2 of [19] for details) with time step $\Delta t = 0.03125$ to obtain the finite-dimensional matrix approximations $\mathbf{Q}_j$, $j = 1, \ldots, 5$. We need to check condition (56), i.e.,

$$\rho\Big(\sum_{j=1}^{5} w_j \mathbf{Q}_j \otimes \mathbf{Q}_j\Big) < 1, \tag{61}$$

for the stability of the second moment of system (57)-(59). Let us pick $(k_1, k_2) = (-1.5, -0.5)$ as an example and look for a region in the (a_1, a_2)-plane where the closed-loop system is stable. In Fig. 2(a) the red curve shows the second moment stability boundary of (57)-(59). For comparison, the blue curve in Fig. 2(b) shows the stability boundary for the same system with a deterministic fixed delay at $\tau = 1$ which is the average of the delay values in the stochastic system. These results demonstrate that the stochasticity in the delay can change the region of stability significantly.

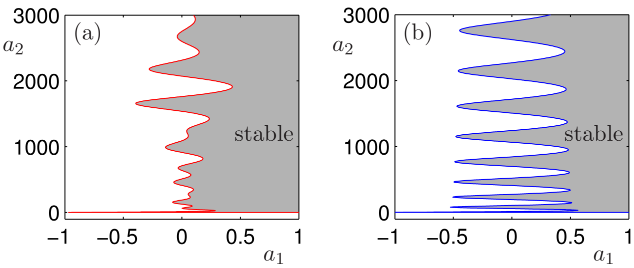

Fig. 2. (a) Stability chart for the stochastic system (57)-(59) when $\tau(t) \in \{0.7, 0.85, 1, 1.15, 1.3\}$, $w_j = \frac{1}{5}$, $\forall j = 1, \ldots, 5$, and $T = 0.5$. (b) Stability chart for the deterministic system (57)-(59) with $\tau(t) = 1$ that is the average of the delay values used in panel (a). The stable areas are shaded.

VI. CONCLUSION

A method for analyzing the stability of linear continuous-time systems with stochastic delay was proposed when the delay could switch between different, but finitely many values stochastically. The system was said to be stable if the second moment, that was defined properly, converged to zero as time went to infinity. Suggested by the form of the delay variations, a linear map was constructed using solution operators that were defined for the stochastic delayed system. Then using the tensor products of linear infinite-dimensional maps the time evolution of the second moment was characterized. The stability criteria were obtained as necessary and sufficient conditions that could not be derived using Lyapunov-based theorems. An example showed that the stochastic delay changed the stability boundaries significantly compared to the deterministic system with the average delay. The model proposed in this paper may be generalized to the case where there are many delays in the system, often referred to as a distributed delay system, while each delay can be stochastic.

ACKNOWLEDGMENT

We thank professor Raymond Ryan for helpful discussion.

REFERENCES

[1] J. Nilsson, B. Bernhardsson, and B. Wittenmark, "Stochastic analysis and control of real-time systems with random time delays," *Automatica*, vol. 34, no. 1, pp. 57–64, 1998.

[2] R. Krtolica, Ü. Özgüner, H. Chan, H. Göktaş, J. Winkelman, and M. Liubakka, "Stability of linear feedback systems with random communication delays," in *Proceedings of the American Control Conference*, 1991, pp. 2648–2653.

[3] W. B. Qin, M. M. Gomez, and G. Orosz, "Stability and frequency response under stochastic communication delays with applications to connected cruise control design," *IEEE Transactions on Intelligent Transportation Systems*, 2016. [Online]. Available: http://dx.doi.org/10.1109/TITS.2016.2574246

[4] F. Bai and H. Krishnan, "Reliability analysis of dsrc wireless communication for vehicle safety applications," in *Intelligent Transportation Systems Conference (ITSC), IEEE*, 2006, pp. 355 – 362.

[5] H. H. McAdams and A. Arkin, "It's a noisy business! genetic regulation at the nanomolar scale," *Trends in Genetics*, vol. 15, no. 2, pp. 65–69, 1999.

[6] K. Josic, J. M. Lopez, W. Ott, L. J. Shiau, and M. Bennet, "Stochastic delay accelerates signaling in gene networks," *PLoS Computational Biology*, vol. 7, no. 11, 2011.

[7] J. Lewis, "Autoinhibition with transcriptional delay: a simple mechanism for the zebrafish somitogenesis oscillator," *Current Biology*, vol. 13, no. 16, pp. 1398–1408, 2003.

[8] M. Sadeghpour and G. Orosz, "On the stability of continuous-time systems with stochastic delay: applications to gene regulatory circuits," in *ASME IDETC*, Buffalo, NY, USA, 2014.

[9] M. M. Gomez, W. B. Qin, G. Orosz, and R. M. Murray, "Exact stability analysis of discrete-time linear systems with stochastic delays," in *American Control Conference (ACC)*, 2014, pp. 5534–5539.

[10] I. Kats, "On the stability of systems with random delay in the first approximation," *Prikladnaya Matematika i Mekhanika (P. M. M.), translated as Journal of Applied Mathematics and Mechanics*, vol. 31, no. 3, pp. 447–452, 1967.

[11] E. A. Lidskii, "On stability of motion of a system with random delays," *Differentialnye Uravnenia, translated as Differential Equations*, vol. 1, no. 1, pp. 96–101, 1965.

[12] I. Kolmanovsky and T. Maizenberg, "Stochastic stability of a class of nonlinear systems with randomly varying time-delay," in *Proceedings of the American Control Conference*, Chicago, Illinois, USA, 2000.

[13] V. L. Kharitonov and S.-I. Niculescu, "On the stability of linear systems with uncertain delay," *IEEE Transactions on Automatic Control*, vol. 48, no. 1, p. 127132, 2003.

[14] Y. Sun and J. Xu, "Stability analysis and controller design for networked control systems with random time delay," in *The Ninth International Conference on Electronic Measurement and Instruments (ICEMI)*, 2009, pp. 136–141.

[15] E. I. Verriest and W. Michiels, "Stability analysis of systems with stochastically varying delays," *Systems and Control Letters*, vol. 58, pp. 783–791, 2009.

[16] J. K. Hale and S. M. V. Lunel, *Introduction to functional differential equations*. Springer-Verlag, 1993.

[17] K. M. Przyluski, "Stability of linear infinite-dimensional systems revisited," *International Journal of Control*, vol. 48, no. 2, pp. 513–523, 1988.

[18] R. A. Ryan, *Introduction to tensor products of Banach spaces*. Springer, 2002.

[19] T. Insperger and G. Stépán, *Semi-discretization for time-delay systems, stability and engineering applications*. Springer, 2011.

[20] D. Breda, "Solution operator approximations for characteristic roots of delay differential equations," *Applied Numerical Mathematics*, vol. 56, no. 3-4, pp. 305–317, 2006.