A Hybrid Vintage Growth Model with
Heterogeneous Capital
-Preliminary and Incomplete-

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Abstract

This paper develops and analyzes a neoclassical vintage growth model that features two distinct types of capital in two aspects; different depreciation rates and different obsolescence. I show that when two types of capital have different depreciation rates and there is no vintage specific technological change: (i) the ratio of stock of capital with lower depreciation rate (A) to capital with higher rate (B) is higher than the ratio of capital shares of A to B; and in a steady state (ii) the ratio of investment of those is lower than the ratio of capital shares; and (iii) a change in rate of neutral technological progress has a negative association with the ratio of stock and a positive association with the ratio of investment.

I show, assuming one of capital is vintage specific and another is not, and that there are both neutral and vintage specific technological changes, that the aggregate production function is the same form of vintage production function with aggregate capital across vintage that is evaluated at market value. I further show that when those two types of capital depreciate at different rates in terms of the net of physical depreciation and obsolescence, depending on the combination of parameters, the model generates unique allocation patterns of new investment and existing capital stock across the two types of capital. In particular, in a steady state: (i) all the new investment goes to vintage non-specific capital, and to vintage specific capital with the newest technology; (ii) a part of the existing stock of vintage non-specific capital is either reinvested or disinvested depending on the relationship of net depreciation rates of capital; and (iii) a change in the rate of vintage specific technological progress, in contrast to a change in the rate of neutral technological progress above, might have either positive or negative associations with the ratio of stock and that of investment.

Assuming that structure is not vintage specific and has a lower depreciation rate, and that equipment is vintage specific and has a higher
depreciation rate, an empirical analysis shows that observed patterns of ratios of stock of structure to equipment and of investment in the U.S. and Japanese aggregate economic data are generally consistent with the proposed model. The analysis further shows that capital shares of structure and equipment both are approximately 0.15, and that vintage specific technological change consists about 60% of multifactor productivity growth of the U.S. economy. (JEL: O)
Figure 1: Nominal Price Changes in Structure, Equipment, Consumption, and GDP, and Price Ratio of Equipment to GDP in the U.S., NIPA, BEA.

1 Introduction

Since Solow’s pioneering growth model [14], economists have kept improving his model in various ways. Empirically, growth accounting based on the model with national account data, such as the U.S. National Income and Product Account (NIPA) [2] and the Japan Industry Productivity (JIP) database [7], is now widely accepted as a valuable application. Although these empirical studies carefully consider heterogeneity of capital that includes differences in depreciation rates and changes in prices of capital, little attention, however, has been paid to the heterogeneity of capital in the model itself. Indeed, there is no analytical study that exactly explains the behavior of the economy when capital is heterogeneous.

This neglect of the heterogeneity of capital raises both theoretical and empirical challenges. A major theoretical challenge is the validity of the assumptions of the model. Typical existing growth models, including vintage growth models, assume a homogeneous obsolescence and a single depreciation rate of physical capital, even though the U.S. national account data shows that this is not the case. Indeed, the data shows that structure has almost no obsolescence and has a lower depreciation rate, whereas equipment has obsolescence and a higher depreciation rate. Figure 1 shows the change in prices of structure, equipment, consumption, and GDP, and the price ratio of equipment to GDP. As can be
seen, rise in equipment prices is slower than the others, and the other three prices show a similar trend. Geometric yearly average growth rates of nominal prices of the structure, equipment, consumption, and GDP between 1960 and 2004 are 4.6%, 1.5%, 3.8%, and 3.8% respectively. Clearly, equipment has a larger drop in the real price due to obsolescence, while structure does not. This is, of course, in line with a usual business expectations that new machines might be relatively important in new production processes, while new industrial buildings might not. As for the depreciation rates, Table 3 of Fraumeni [6] reports that the depreciation rates of structure are lower than those of equipment on average; those of private nonresidential structure range from 1.6% to 7.5%, whereas those of private nonresidential equipment range from 5% to 31%.

A major empirical challenge to homogeneous capital model, for example, is that the observed relationships of ratios of stock of structure ($A$) to equipment ($B$), $A/B$, and of investment, $I^A/I^B$, cannot be explained by existing growth models that assume homogeneity of physical capital. Figure 2 shows those ratios in the U.S. data. On the one hand, as can be seen in the figure, the data shows that the ratio of stock of structure to equipment, $A/B$ is larger than that of investment, $I^A/I^B$. On the other hand, Shell and Stiglitz’s two capital model [13], which assumes a single depreciation rates of two types of

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1These depreciation rates include both physical depreciation and obsolescence as pointed out by Fraumeni. This issue will be discussed in a later section.

2We observe the same relationships in the Japanese data. See Figure 5.
non-vintage-specific capital, expects $A/B = I^A/I^B = \alpha/\beta$, where $\alpha$ and $\beta$ are capital shares of two types of capital. This result is robust in a vintage growth model by Laitner and Stolyarov [12] that assumes a single depreciation rate and a homogeneous obsolescence of the two types of capital.

In order to address these issues, this paper develops and analyzes a neo-classical vintage growth model with heterogeneous capital. As the above data and the preceding argument might suggest, we assume that one capital is vintage specific and the other is not, and that two types of capital have different depreciation rates. Studies that have a similar setup include Greenwood, Hercowitz, and Krusell (GHK) [9] and Aruga [1]. Our study is different from GHK because the current model has an analytical solution for both transition and steady state based on a Solow style constant saving rate, while GHK’s does not. Aruga studies a vintage capital growth model that features intangible capital and tangible capital with different depreciation rates. Aruga assumes that both types of capital are vintage specific, while the current model assumes only one type of capital is vintage specific.

A theoretical analysis of a disembodied model shows, assuming there is no vintage specific technological change and two types of capital depreciate at different rates, that: (i) the ratio of stock of capital with lower depreciation rate ($A$) to capital with higher rate ($B$) is higher than the ratio of capital shares of $A$ to $B$; and in a steady state (ii) the ratio of investment of those is lower than the ratio of capital shares; and (iii) a change in rate of neutral technological progress has a negative association with the ratio of stock and a positive association with the ratio of investment.

A theoretical analysis of a hybrid vintage growth model, which assumes one of capital is vintage specific and another is not, and there are both neutral and vintage specific technological changes, provides a determination mechanism of the price of vintage specific capital, and the allocation of investment and stock across the two types and across capital vintage. I show that the aggregate production function is the same form of vintage production function with aggregate capital across vintage that is evaluated at market value. I further show that when these two types of capital depreciate at different rates in terms of the net of physical depreciation and obsolescence, depending on the combination of parameters, the model generates unique allocation patterns of new investment and existing capital stock across the two types of capital. In particular, in a steady state: (i) all the new investment goes to vintage non-specific capital, and to vintage specific capital with the newest technology; (ii) a part of the existing stock of vintage non-specific capital is either reinvested or disinvested depending on the relationship of net depreciation rates of capital; and (iii) a change in the rate of vintage specific technological progress, in contrast to a change in the rate of neutral technological progress above, might have either positive or negative associations with the ratio of stock and that of investment.

We apply the model to actual economy by assuming that structure is not vintage specific and has a lower depreciation rate, while equipment is vintage specific and has a higher depreciation rate. An empirical analysis with U.S. and Japanese tangible wealth data from BEA [3] and JIP database shows that
this model is generally consistent with observed levels of the ratios of stock and investment in both U.S. and Japanese data. Based on those levels, we also estimate that capital shares are $\alpha = \beta = 0.15$ based on NIPA data in conjunction with Gordon’s improved price index [8]. Furthermore, we demonstrate that observed changes in ratios of investment and stock of capital in the U.S. can be explained by the proposed model if the bias in NIPA equipment prices is taken into account. A growth accounting based on the model shows that about 60% of multifactor productivity growth is attributed to vintage specific technological change.

The structure of the rest of the paper is as follows: Section 2 develops dis-embodied model, Section 3 develops hybrid model, Section 4 analyzes steady state, Section 5 demonstrates empirical analysis, Section 6 discusses the result, and Section 7 concludes the paper.

2 Dis-embodied Model

2.1 Basic Setup

In the first part of the paper, I consider the case where there are two types of capital and neither of them embodies vintage specific technological change. Aggregate production function is simply assumed to be a form of Cobb-Douglas:

$$ Y(t) = z(t)A(t)^\alpha B(t)^\beta L(t)^{1-\alpha-\beta}, \quad (1) $$

where $z(t)$ is the Hicks neutral level of vintage non-specific technology and monotonically increasing in $t$, $A(t)$ and $B(t)$ are the stocks of different capital, and $L(t)$ is the level of labor supply. Assume that the growth rates of the technology and the labor are always positive. The capital shares, $\alpha \in (0, 1)$, and $\beta \in (0, 1)$ where $\alpha + \beta < 1$ are assumed to be constant. The production function exhibits constant returns to scale in $A(t), B(t)$, and $L(t)$, strictly increasing in each factor, and strictly decreasing returns in each factor.

Assume the factors markets and the final goods markets are competitive, and capital $A$ and $B$ have different depreciation rates, $\delta^A$ and $\delta^B$ where $\delta^A < \delta^B$. Assume further that agents have perfect foresight and are rational, and investment is irreversible. Then the profit maximization conditions of the economy are

$$ MPA(t) = \alpha \frac{Y(t)}{A(t)} = P^A(t) \left[ r(t) + \delta^A - \dot{P}^A(t) \right] \quad (2) $$

$$ MPB(t) = \beta \frac{Y(t)}{B(t)} = P^B(t) \left[ r(t) + \delta^B - \dot{P}^B(t) \right] \quad (3) $$

$$ MPL(t) = (1 - \alpha - \beta) \frac{Y(t)}{L(t)} = W(t) \quad (4) $$

where $r(t)$ is the real interest rate, $P^A(t)$ and $P^B(t)$ are the prices of capital $A$ and $B$, and $W(t)$ is the wage. Note that $\hat{}$ denotes time derivative of natural
logarithm of argument throughout in this paper. The setups are very similar to those of Shell and Stiglitz [13] except for that capital A and B have different depreciation rates here.

Assume output can be used to either consumption, capital A investment, and/or capital B investment, and the economy uses fixed share \( \sigma \in (0, 1) \) of the output for the investment. Then, allocation of investment is simply

\[
\sigma Y(t) = I^A(t) + I^B(t).
\]

### 2.2 Dynamics

Shell and Stiglitz show that, when \( \delta^A = \delta^B \), given initial endowments of two kinds of capital \( A(0) \) and \( B(0) \), and assuming the short-run perfect foresight, there is a unique assignment of asset prices and an allocation of investment that lead to long-run balanced growth that is *intertemporally efficient*. The same argument can apply in the current model with some modification. To see it, let’s cancel the interest rate \( r(t) \) in (2) and (3):

\[
\beta P_B(t) B(t) - \alpha P_A(t) A(t) = \left( \delta^B - \hat{P}^B(t) \right) - \left( \delta^A - \hat{P}^A(t) \right).
\]

Now, in order to make the analysis clear, define effective labor:

\[
E(t) \equiv z(t)^{1/(1-\alpha-\beta)} L(t),
\]

and use lower case letters to express per effective labor amount:

\[
y(t) \equiv \frac{Y(t)}{E(t)}, a(t) \equiv \frac{A(t)}{E(t)}, \text{ and } b(t) \equiv \frac{B(t)}{E(t)}.
\]

Then, (5) becomes

\[
\beta a(t)^\alpha b(t)^{\beta-1} - \alpha a(t)^{\alpha-1} b(t)^{\beta} = \left[ \delta^B - \hat{P}^B(t) \right] - \left[ \delta^A - \hat{P}^A(t) \right].
\]

The following analysis about the dynamics of the economy is very similar to that of Shell and Stiglitz [13]. Their equation (2.6) can be replaced with (6) of the current model and we can follow their argument to determine the dynamics of the economy. Their straight line \( OA \) in the Figure II is replaced with

\[
\beta a(t)^\alpha b(t)^{\beta-1} - \alpha a(t)^{\alpha-1} b(t)^{\beta} = \delta^B - \delta^A,
\]

where on the curve there is investment in both A and B capital and prices of them are one.

From (7), we can derive following proposition easily.

**Proposition 1 (Ratio of Stock)** Assume the economy invests in both capital A and B, and \( \delta^A < \delta^B \) without loss of generality. Then, the relationship between the ratio of stock of those capital, \( A(t)/B(t) \), and the ratio of shares of those, \( \alpha, \beta \) is:

\[
\frac{\alpha}{\beta} < \frac{A(t)}{B(t)}
\]
Proof. See Appendix.

Intuitively, since the depreciation rate of structure is smaller than that of equipment, holding structure is cheaper than that of equipment and economy has more structure compared to equipment.

Here, \( a(t) \) is increasing and convex in \( b(t) \) when \( a(t) > 0 \) and \( b(t) > 0 \) as in Figure 3. So, if we want to keep investing in both capital, we have to decide investment such that the stocks are on this curve, because otherwise at least one of prices should depart from one. As argued in Shell and Stiglitz, the economy should follow on this path once the economy reach on it in order to converge to a steady state.

If the economy starts from a point \((t = 0)\) below \((7)\) in Figure 3, then, firms invest only in capital \(A\) and the relationship of prices is \(P^A(t) = 1 > P^B(t)\). It should be observed from \((6)\) that we can choose \(P^B(0)\) such that \(P^B(t) > 0\) and \(P^B(t)\) reaches one when the economy reaches \((t = 1)\) in order to the economy be \textit{intertemporally efficient} as in the Shell and Stiglitz. Then, economy converges to the steady state (S.S.). A steady state that has positive output must be on the curve \((7)\) because otherwise the economy’s investment specializes in one of the capital, which cannot be the desirable steady state.

\[\alpha/\beta\]

\[t=0\]

\[t=1\]

\[S.S.\]

\[O\]

\[a(t)\]

\[b(t)\]

\[\text{Equation (8)}\]

Figure 3: Per Effective Amounts of \(a(t)\) and \(b(t)\).

\[3\]Differentiate both sides of \((7)\) by \(b\). Then, \[\frac{da}{db} = \frac{\alpha(1-\beta)(\beta-1) a}{\alpha^2 - \alpha(1-\beta)(\beta-1)b} > 0, \] and \[\frac{d^2a}{db^2} = \frac{\alpha(1-\beta)(\beta-1) [\alpha \beta - \alpha(1-\beta)]}{\alpha^2 - \alpha(1-\beta)(\beta-1)b} > 0\] because \((7)\) implies \(\beta a - \alpha b > 0\).
Laws of motion of the two kinds of capital is written as
\[ \dot{A}(t) = I^A(t) - \delta^A A(t), \]
\[ \dot{B}(t) = I^B(t) - \delta^B B(t). \]
(8)

If the economy is not on (7), the investment specializes one of the capital, and otherwise, it should follow the time differential of (7),
\[ \dot{a}(t) = \frac{\dot{A}(t) - A(t)\dot{E}(t)}{B(t) - B(t)\dot{E}(t)} = \frac{I^A(t) - [\delta^A + \dot{E}(t)]A(t)}{I^B(t) - [\delta^B + \dot{E}(t)]B(t)}. \]
(10)

2.3 Steady State

Now, the relationship (7) can be written as
\[ a(t) = f(b(t)). \]
(11)

Note that as in footnote 5,
\[ f'(b) > 0. \]
(12)

On the other hand, sum of the laws of motion with the per effective capita expression (8) and (9) is
\[ \dot{a}(t) + \dot{b}(t) = \sigma y(t) - \delta^A a(t) - \delta^B b(t) - \dot{E}(t)[a(t) + b(t)]. \]
(13)

Since (11) implies \( \dot{a}(t) = f'(b(t))\dot{b}(t) \), (13) can be rewritten as
\[ \dot{b}(t) = \frac{\sigma f(b(t))^\alpha b(t)^\beta - \delta^A f(b(t)) - \delta^B f(b(t)) - \dot{E}(t)[f(b(t)) + b(t)]}{f'(b(t)) + 1}. \]
(14)

Clearly, \( \dot{b}(t) = 0 \) when \( b = 0 \). In addition, when the growth rates of technology and labor are constant, the sufficient condition of having positive steady state \( b^* > 0 \) where \( \dot{b}(t) = 0 \) is
\[ 0 < \sigma < \frac{\delta^B}{\delta^A - \delta^B \beta}, \]
(15)

which implies that there is no steady state equilibrium when depreciation rate of capital \( A \) is high enough compared to the sum of obsolescence and physical depreciation rate of capital \( B \). \footnote{To see it, observe that the Inada condition holds, and \( \lim_{b \to \infty} (b/a) = 0 \) and thus numerator of (14) \( \frac{[\sigma f(b)^\alpha b^\beta - \delta^B f(b^\alpha \beta)] - \delta^A f(b^\alpha \beta) - \dot{E}(1+b^\alpha \beta)}{\beta - \alpha (b/a)} \) becomes negative as \( b \to \infty \) when (15) holds.} In other words, a divergence of economy is possible with certain set of parameters that is not allowed in the basic Solow model. Intuitively, this is because when the saving rate \( \sigma \) is too high, investment
cannot be below the net depreciation because $g(t)$ is not globally concave in $b(t)$. At $b^*$, (12) implies $a^* > 0$ and $\dot{a}(t) = 0$, and the economy is stable.\footnote{Observe that the first series of the Taylor approximation of the summarized law of motion of capital (14) is $b(t) \approx \frac{\alpha + \beta - 1}{1 - \alpha - \beta} \{ \delta A + \Delta L(t) (a^*/b^*) + \alpha (\delta A + \Delta L(t) (a^*/b^*)) \} (b(t) - b^*)$, where the coefficient is negative.}

Assuming (15), the full characterization of a steady state is given in the following proposition.

**Proposition 2** Assume the growth rate of labor is constant, $\hat{\ell}$, the growth rate of technology is constant, $\hat{z}$, and parameters satisfy (15). Assume further that the economy is in a steady state. Then, given the steady state value $a^*$ and $b^*$ from (11) and (14) and the current $L(t)$ and $z(t)$, the economy is characterized as following:

1. **Capital Stock**
   
   \[ A(t) = a^* E(t) \]
   \[ B(t) = b^* E(t) \]

2. **Prices of Capital**
   
   \[ P^A(t) = P^B(t) = 1 \]

3. **Allocation of investment**
   
   \[ I^A(t) = \left( \delta^A + \hat{E} \right) A(t) \] \tag{16}
   \[ I^B(t) = \left( \delta^B + \hat{E} \right) B(t) \] \tag{17}

   where $\hat{E} = \hat{E}(t) = \frac{\hat{z}}{1 - \alpha - \beta} + \hat{L}$.

   **Proof.** See Appendix.

   So, in a steady state, economy invests in both types of capital, prices of capital are 1, $A(t)$ and $B(t)$ grow at the rate of effective labor growth, and $I^A(t)$ and $I^B(t)$ are proportional to $A(t)$ and $B(t)$ as in the basic Solow model. Differences from the Solow’s model are simply from the difference in $a^*$ and $b^*$.

   It is useful to derive several comparative statics result according to this model.

**Proposition 3 (Ratio of Investment)** Assume the economy is in a steady state, which is characterized by the above proposition, and $\delta^A < \delta^B$ without loss of generality. Then, the relationship between the ratio of investment of those capital, $I^A(t)/I^B(t)$, and the ratio of shares of those, $\alpha/\beta$ is

\[ \frac{I^A(t)}{I^B(t)} < \frac{\alpha}{\beta}. \] \tag{18}
Proof. See Appendix.

The investment of structure is smaller than that of equipment in a steady state because otherwise the ratio $A/B$ cannot be constant. Clearly, combined with Proposition 2, the result $I^A(t)/I^B(t) < A(t)/B(t)$ is consistent with data as in the introduction.

Another possibly useful comparative statics is the changes in those ratios in response to the change in the neutral rate of technological progress.

**Proposition 4 (Change in Ratio of Stock and Investment when $\hat{z}$ changes)**

Assume the economy is in a steady state, which is characterized by the above proposition, and $\delta^A < \delta^B$ without loss of generality. Then, if the saving rate $\sigma$ is below the Golden Rule, $\sigma < \alpha + \beta$, a change in the rate of technological progress, $\hat{z}$, shifts the steady state equilibrium as following:

$$\frac{d}{d\hat{z}} \left[ \frac{A(t)}{B(t)} \right] \leq 0 \leq \frac{d}{d\hat{z}} \left[ \frac{I^A(t)}{I^B(t)} \right].$$

(19)

Proof. See Appendix.

The result about stock is easily confirmed in Figure 3. When the rate of technological progress, $\hat{z}$, is high, as in the basic Solow model, we have lower per effective capita amount, $a^*$ and $b^*$, while they are higher when $\hat{z}$ is low. As can be seen in the figure, higher $a^*$ and $b^*$ when $\hat{z}$ is low result in a higher ratio, $a^*/b^* = A(t)/B(t)$.

Intuitively, when the technological progress is slow, interest rate is low. Then, capital with lower depreciation rate is more attractive because it has longer life, which explains the left hand side of (19). On the other hand, ratio of investment is affected by the change in $(\delta^A + \dot{E})/(\delta^B + \dot{E})$, which is positively associated with $\hat{z}$. The result is the right hand side of (19).

3 Hybrid Model

3.1 Basic Setup

We consider the case where one of capital is vintage specific and the other is not: a hybrid vintage model. We assume two kinds of capital, $A$ and $B$. Each $B$ capital is vintage specific and $A$ is not, which means that reallocation of $A$ capital has no restriction. Let’s $v$ denote the vintage of technology: e.g., $A_v(t)$ represents the physical amount of capital $A$ that works for vintage technology $v$ production at time $t$. Similarly, $B_v(t)$ and $L_v(t)$ represent the number of units of capital $B$ and labor for vintage $v$ at time $t$. Assume the production function with vintage technology $v$ at time $t$ is standard Cobb-Douglas

$$Y_v(t) = z(t)q_vA_v(t)\alpha B_v(t)\beta L_v(t)^{1-\alpha-\beta}.$$  

(20)
Note that there are two sources of technological change. One is neutral technological progress, $z(t)$, which is not vintage specific. The other is vintage specific technological progress, $q_v$. Technology level, $z(t)$ and $q_v$ are monotonically increasing in $t$ and $v$ respectively. Assume capital $A$ and $B$ physically depreciate at the rates $\delta^A$ and $\delta^B$. Further assume that agents have perfect foresight and are rational, and investment is irreversible. The profit maximization conditions are

$$MPA(t) = \alpha \frac{Y_v(t)}{A_v(t)} = P^A(t)[r(t) + \delta^A - \hat{P}^A(t)], \quad (21)$$

$$MPB_v(t) = \beta \frac{Y_v(t)}{B_v(t)} = P^B_v(t)[r(t) + \delta^B - \hat{P}^B_v(t)], \quad (22)$$

$$MPL(t) = (1 - \alpha - \beta) \frac{Y_v(t)}{L_v(t)} = W(t), \quad (23)$$

where $\hat{\text{hat}}$ denotes the time derivative of natural log of the argument.

Note that, the price of capital $A$ is independent of vintage because capital $A$ can be freely reallocated across vintage, and hence that the $MPA(t)$ is also independent of vintage subscript. This is the main difference from Aruga [1]. Further note that the prices of capital have to satisfy $P^A(t) \in [0, 1]$ and $P^B_v(t) \in [0, 1]$, where the lower bound and the upper bound are from the assumptions that capital is freely disposable and investment in vintage capital, $v$, is possible at anytime $t \geq v$ respectively.

Aggregate investment is again Solow type:

$$\sigma Y(t) = I^A(t) + I^B(t)$$

$$= I^A_t(t) + \int_{-\infty}^{T} I^A_v(t)dv + I^B(t), \quad (24)$$

where $T$ denotes the frontier technology and $Y(t)$ is the aggregate production defined by (33).\(^6\) Note that the price of investment goods is 1 at time $t$ in terms of output at time $t$ no matter for what vintage we invest in.

### 3.2 Aggregation

The aggregation method here is similar to that of Aruga. Compare the vintage production $v$ and $\bar{v}$ at time $t \geq v, \bar{v}$. We have two vintage production function, (20) and

$$Y_v(t) = z(t)q_vA_v(t)^\alpha B_v(t)^\beta L_v(t)^{1-\alpha-\beta}. \quad (25)$$

Then, using (21), (22), and (23), get

$$\left[ \frac{MPB_v(t)}{MPB_{\bar{v}}(t)} \right]^{\beta} = \frac{q_v}{q_{\bar{v}}}, \quad (26)$$

\(^6\)I use $T$ for the frontier capital instead of $t$ to avoid complication of time and frontier technology vintage.
This equation summarizes the profit maximization conditions by the firms.

Now, consider the relationships between a vintage, \( v \), and the frontier vintage, \( T \). Applying (21) - (23) to \( v = v \) and \( \bar{v} = T \), read

\[
\frac{A_v(t)}{L_v(t)} = \frac{A_T(t)}{L_T(t)}, \quad (27)
\]

\[
\frac{MPB_v(t) B_v(t)}{MPB_T(t) L_v(t)} = \frac{B_T(t)}{L_T(t)}. \quad (28)
\]

Note that the right hand side of (27) and (28) is valid only if \( A_T(t) \) and \( B_T(t) \) exists. I here call that those are the implicit capital if the capital with frontier technology had existed. Then, define aggregate capital,

\[
A(t) = \int_{-\infty}^{T} A_v(t) dv = \frac{A_T(t)}{L_T(t)} L(t), \quad (29)
\]

\[
B(t) = \int_{-\infty}^{T} \frac{MPB_v(t)}{MPB_T(t)} B_v(t) dv = \frac{B_T(t)}{L_T(t)} L(t), \quad (30)
\]

where \( L(t) \) is defined as aggregate labor by following:

\[
L(t) = \int_{-\infty}^{T} L_v(t) dv. \quad (31)
\]

Note that if return on capital, \( R_B^P(t) \) are independent of vintage \( v \) respectively, (30) simply shows conventional market value of capital evaluated at output price. These equation implies that the aggregate capital can be summarized by the implicit frontier capital per labor, \( A_T(t)/L_T(t) \) and \( B_T(t)/L_T(t) \). As can be seen below, this definition of aggregate capital is useful to show the aggregate production function. Further observe that the implicit ratios of frontier capital and the aggregate capital are the same:

\[
\frac{A(t)}{B(t)} = \frac{A_T(t)}{B_T(t)}. \quad (32)
\]
Using (20), and (26) - (32), the aggregate output can be rewritten as

\[ Y(t) = \int_{-\infty}^{T} Y_v(t)dv \]

\[ = \int_{-\infty}^{T} z(t)q_vA_v(t)A(t)B_v(t)B(t)\alpha L_v(t)\beta L(t)\gamma 1^{-\alpha-\beta}dv \]

\[ = z(t)\int_{-\infty}^{T} \left[ A_T(t) \right]^{\alpha} \left[ B_T(t) \right]^{\beta} \int_{-\infty}^{T} \left[ q_vB_v(t)M_T(t) \right]^{\beta} L_v(t)dv \]

\[ = z(t)q_T\int_{-\infty}^{T} A(t)A(t)B(t)B(t)B(t)\alpha L(t)\beta L(t)1^{-\alpha-\beta}. \]

Surprisingly enough, aggregate production function is the same form as (20) with frontier vintage specific technology level \( q_T \) and the aggregate capital defined as (29) and (30). Note that we don’t have to specify the prices of capital \( A \) and \( B \) in order to derive the aggregate production function since we define the aggregation of the capital by (29) and (30). [Cite Fisher??]

There is another useful way to express the aggregate capital and aggregate production function. From (26) and (30), we have

\[ B(t) = \int_{-\infty}^{T} \left[ q_T \right]^{-\beta} B_v(t)dv \]

\[ = q_T^{-\beta} \int_{-\infty}^{T} B_v(t)B(t)dv \]

\[ = q_T^{-\beta} J(t), \]

(34)

where

\[ J(t) = \int_{-\infty}^{T} [q_vB_v(t)\beta]^{1/2} dv. \]

(35)

(34) can be rearranged as

\[ J(t)^{\beta} = q_T B(t)^{\beta} \]

(36)

and therefore, the aggregate production (33) is shown as

\[ Y(t) = z(t)A(t)J(t)^{\beta} L(t)1^{-\alpha-\beta}. \]

(37)

Interestingly enough, this aggregation is the same form as Solow [15]. \( J(t) \) stands for Solow’s Jelly Capital.
3.3 Price Distribution of Vintage Specific Capital $B$

Next, we determine the price distribution of capital $B$. Note that wage and the price of capital $A$ are unique across vintage because both labor and capital $A$ is freely allocated across vintage. Now, in order to obtain the price of $B$ capital, rewrite (26) as

$$P_B^v(t)\{r(t) + \delta^B\} - \bar{P}_B^v(t) = \left[\frac{q_v}{q_v}\right]^{-\frac{1}{\eta}} \left[\bar{P}_B^v(t)\{r(t) + \delta^B\} - \bar{P}_B^v(t)\right],$$

where \(\dot{}\) denotes the time derivative of the argument. By multiplying both sides by \(e^{-\frac{1}{\eta} \int_{-\infty}^{t} r(u) du + \delta^B t}\), and integrating over $t$, read

$$\int_{-\infty}^{t} e^{-\frac{1}{\eta} \int_{-\infty}^{u} r(u) du + \delta^B u} P_B^v(t) du = \left[\frac{q_v}{q_v}\right]^{-\frac{1}{\eta}} \int_{-\infty}^{t} e^{-\frac{1}{\eta} \int_{-\infty}^{u} r(u) du + \delta^B u} \bar{P}_B^v(t) du + C$$

which is equivalent to

$$e^{-\frac{1}{\eta} \int_{-\infty}^{t} r(u) du + \delta^B t} P_B^v(t) = \left[\frac{q_v}{q_v}\right]^{-\frac{1}{\eta}} e^{-\frac{1}{\eta} \int_{-\infty}^{t} r(u) du + \delta^B t} \bar{P}_B^v(t) + C$$

where $C$ is some constant. But since this should be true for $\bar{v} = v$, we have $C = 0$. Therefore, the price of capital $B$ is

$$P_B^v(t) = \left[\frac{q_v}{q_v}\right]^{-\frac{1}{\eta}} \bar{P}_B^v(t).$$

This price can be related to the implicit price of frontier capital $B$, $P_B^T(t)$, by applying to $\bar{v} \rightarrow T$:

$$P_B^v(t) = \left[\frac{q_T}{q_v}\right]^{-\frac{1}{\eta}} \bar{P}_B^T(t). \quad (38)$$

This equation implies that there is no investment in capital $B$ with vintage $v < T$ because the prices of them are always less than one.

Note that since the change in price of capital $B$, $P_B^v(t)$ is independent of vintage $v$, (30) can be shown as

$$B(t) = \int_{-\infty}^{T} \frac{P_B^v(t)}{P_B^T(t)} \frac{R_B^v(t)}{R_B^T(t)} B_v(t) dv = \int_{-\infty}^{T} \frac{P_B^v(t)}{P_B^T(t)} B_v(t) dv, \quad (39)$$

which implies the aggregate $B$ capital is simply the aggregate value of vintage $B$ capital in terms of price of frontier vintage $B$ capital.

In particular, when there is investment in $B$ capital, we have $P_B^T(t) = 1$ and (39) simply becomes the aggregate market value of vintage $B$ capital, and we have

$$P_B^v(t) = \left[\frac{q_T}{q_v}\right]^{-\frac{1}{\eta}}. \quad (40)$$
3.4 Allocation of Capital

The relationship between the marginal products of capital and price of capital pins down the allocation of capital in the competitive market. By canceling the interest rate, \( r(t) \), from (21), (22), and using (38), we have

\[
\left[ \frac{\beta}{P_v(t)B_v(t)} - \frac{\alpha}{P^A(t)A_v(t)} \right] Y_v(t) = [\delta^B(t) - \hat{P}^B(t)] - [\delta^A - \hat{P}^A(t)], \tag{41}
\]

where \( \delta^B(t) \), the net depreciation rate of vintage specific capital \( B \), can be defined as the sum of physical depreciation, and obsolescence,

\[
\hat{\delta}^B(t) = \delta^B + \hat{q}_t. \tag{42}
\]

Now, define the effective labor for vintage \( v \): \( E_v(t) = [z(t)q_v]^{1/(1-\alpha-\beta)}L_v(t) \). Use the lower case letters to express per effective capita amount, \( a_v(t) = A_v(t)/E_v(t), \) \( b_v(t) = B_v(t)/E_v(t) \), and \( y_v(t) = Y_v(t)/E_v(t) \), which implies \( y_v(t) = a_v(t)^{\alpha}b_v(t)^{\beta} \). Then, using the per effective capita amount and (38), we can rewrite (41) as

\[
\frac{\beta a_v(t)^{\alpha}b_v(t)^{\beta-1}}{P_v^B(t)} - \frac{\alpha a_v(t)^{\alpha-1}b_v(t)^{\beta}}{P^A(t)} = [\delta^B(t) - \hat{P}^B(t)] - [\delta^A - \hat{P}^A(t)]. \tag{43}
\]

In order to link the implicit frontier capital to the aggregate capital, further define the aggregate effective labor: \( E(t) = [z(t)q_T]^{1/(1-\alpha-\beta)}L(t) \). Use lower case letters to express per effective capita amount: \( a(t) = A(t)/E(t), \) \( b(t) = B(t)/E(t) \), and \( y(t) = Y(t)/E(t) \), which again implies \( y(t) = a(t)^{\alpha}b(t)^{\beta} \). Then, (29) and (30) imply

\[
a(t) = a_T(t), \tag{44}
\]
\[
b(t) = b_T(t), \tag{45}
\]
\[
y(t) = y_T(t). \tag{46}
\]

where \( a_T(t), b_T(t), \) and \( y_T(t) \) are implicit frontier per effective capita capital and output.

Applying (43) to \( v = T \), we can obtain the aggregate allocation between \( a(t) \) and \( b(t) \):

\[
\frac{\beta a(t)^{\alpha}b(t)^{\beta-1}}{P^B_T(t)} - \frac{\alpha a(t)^{\alpha-1}b(t)^{\beta}}{P^A(t)} = [\delta^B(t) - \hat{P}^B(t)] - [\delta^A - \hat{P}^A(t)]. \tag{47}
\]

So, the relationships of aggregate allocation of \( a(t) \) and \( b(t) \) and change in prices are characterized by (47) and they are the same as the implicit frontier investment allocation. In other words, the implicit frontier allocation reflects overall allocation of capital.

The following analysis is very similar to the disembodied model in Section 2. When investment in both \( A \) and \( B \) occur, since prices are one,

\[
\beta a(t)^{\alpha}b(t)^{\beta-1} - \alpha a(t)^{\alpha-1}b(t)^{\beta} = \delta^B(t) - \delta^A, \tag{48}
\]

16
which is shown in the Figure 4. The major difference from Section 2 is $\hat{\delta}(t)$, which is not constant here whereas this was constant in Section 2. Note that the sign of the right hand side of (48) depends on the relationship among $\delta^A$, $\hat{q}_t$, and $\beta$\textsuperscript{7}. Further note that Proposition 1 is valid. Here, we consider the case where

$$\hat{\delta}^B(t) > \delta^B > \delta^A$$

because we should assume $A$ is structure and $B$ is equipment from the observation of the actual data. Then, the right hand side of (48) is positive, the curve is convex and above the straight line $\alpha/\beta$ under the assumption, and the curve shifts up as the vintage specific technological progress, $\hat{q}_t$, goes up.

Note that when the capital $B$ is also vintage non-specific and there is investment in $B$ capital, $P^B_v(t) = 1 \forall v$ and therefore $\hat{\delta}^B = \delta^B$. This means, if both capital is not vintage specific as in Section 2, economy is on the curve that is independent of rate of vintage specific technological progress, $\hat{q}_t$. If one of the capital is vintage specific and another is not, economy is on the curve that moves depending on the rate of vintage specific technological progress, $\hat{q}_t$.

Further note that even when the depreciation rates are the same, $\delta^A = \delta^B$, like in the Shell and Stiglitz, or Laitner and Stolyarov, the right hand side of (48) is positive in the current hybrid model due to the existence of obsolescence, $\hat{q}_t/\beta$. And thus the dynamics and steady state are different from their models.

\textsuperscript{7}Note that $a(t)/b(t) \geq \alpha/\beta$ if $\delta^B(t) - \delta^A \geq 0$ and $a(t)/b(t) < \alpha/\beta$ otherwise.
3.5 Capital Accumulation when the Economy on the Path

Now, consider the laws of motion of capital. First, the laws of motion of each vintage are given by

\[ \dot{A}_v(t) = I^A_v(t) - \delta^A A_v(t), \]  
\[ \dot{B}_v(t) = -\delta^B B_v(t). \]  

(49)  

(50)

Recall that there is no investment in vintage \( B \) capital as discussed above.

Next, aggregate law of motion of capital is obtained by differentiating (29):

\[ \dot{A}(t) = \frac{\partial}{\partial t} \int_{-\infty}^{t} A_v(t) dv \]
\[ = \int_{-\infty}^{t} \frac{\partial}{\partial t} A_v(t) dv + A_t(t) \]
\[ = \int_{-\infty}^{t} A_v(t) \dot{A}(t) dv + A_t(t) \]
\[ = \int_{-\infty}^{t} A_v(t) \left[ \frac{I^A_v(t)}{A_v(t)} - \delta^A \right] dv + A_t(t) \]
\[ = \int_{-\infty}^{t} I^A_v(t) - \delta^A A(t) + I^A_t(t). \]  

(51)

Similarly, differentiation of (39) gives

\[ \dot{B}(t) = \frac{\partial}{\partial t} \int_{-\infty}^{t} P_v(t) B_v(t) dv \]
\[ = \int_{-\infty}^{t} \frac{\partial}{\partial t} [P_v(t) B_v(t)] dv + P_t(t) B_t(t) \]
\[ = \int_{-\infty}^{t} [P_v(t) B_v(t)] [\dot{P}_v(t) + \dot{B}_v(t)] dv + B_t(t) \]
\[ = -\left[ \frac{\dot{q}}{\beta} + \delta^B \right] B(t) + I^B_t(t) \]
\[ = -\delta^B(t) B(t) + I^B_t(t), \]  

(52)

since there is no investment in vintage \( B \) capital.

The per effective capita expressions are

\[ \dot{a}(t) = \left[ \frac{\dot{A}(t)}{E(t)} \right] \]
\[ = \frac{\dot{A}(t)}{E(t)} - \frac{A(t) \dot{E}(t)}{E(t) E(t)} \]
\[ = \frac{I^A(t)}{E(t)} - [\delta^A + \dot{E}(t)] a(t), \]  

(53)
and similarly
\[ \dot{b}(t) = \frac{I^B(t)}{E(t)} - [\dot{\delta}^B(t) + \dot{E}(t)]b(t). \] (54)

Those can be simplified by using (24) such that
\[ \dot{a}(t) + \dot{b}(t) = \sigma a(t)\beta b(t)^\beta - \delta^A a(t) - \delta^B b(t) - \dot{E}(t)[a(t) + b(t)]. \] (55)

Finally, by using notation \( a(t) = h(b(t), \hat{q}_t) \) for (47), this can be expressed only by \( b(t) \):
\[ \dot{b}(t) = \frac{1}{h_1(b(t), \hat{q}_t) + 1} \cdot \left[ \sigma h(b(t), \hat{q}_t)\alpha b(t)^\alpha - \delta^A h(b(t), \hat{q}_t) - \delta^B b(t) - \dot{E}(t)[h(b(t), \hat{q}_t) + b(t)] - h_2(b(t), \hat{q}_t)\dot{q}_t \right]. \] (56)

Now, let’s consider the allocation of investment. From (21) and (22), we have
\[ \frac{P_v(t)B_v(t)}{A_v(t)} = \frac{B_v(t)}{A_v(t)}. \] (57)

Using (32), (38), (49), (50), and (57), observe
\[ \frac{I^A_v(t)}{A_v(t)} = \dot{A}_v(t) + \delta^A \]
\[ = \delta^A - \delta^B(t) + \dot{A}(t) - \dot{B}(t). \] (58)

So, aggregate investment in vintage \( A \) capital is
\[ \int_{-\infty}^t I^A_v(t)dv = [\delta^A - \delta^B(t) + \dot{A}(t) - \dot{B}(t)]A(t). \] (59)

Allocation of frontier investment, \( I^A_f(t) \) and \( I^B_f(t) \) are given by (51) and (59):
\[ \frac{I^A_f(t)}{I^B_f(t)} = \frac{A_f(t)}{B_f(t)} = \frac{A(t)}{B(t)}, \]
which is consistent with the allocation of aggregate capital.

The dynamics of the economy with 6 unknowns, \( Y(t), A(t), B(t), I^A_f(t), I^B_f(t), \int_{-\infty}^t I^A_v(t)dv, I^B_v(t) \), can be determined by 6 equations, (24), (33), (51), (52), (59), and differential of \( a(t) = h(b(t), \hat{q}_t) \),
\[ \frac{\dot{A}(t)}{E(t)} - \frac{A(t)}{E(t)} \dot{E}(t) = \dot{a}(t) \]
\[ = h_1(b(t), \hat{q}_t)\dot{b}(t) + h_2(b(t), \hat{q}_t)\dot{q}_t \]
\[ = h_1 \left( \frac{B(t)}{E(t)} \right) \dot{q}_t \left[ \frac{B(t)}{E(t)} - \frac{B(t)}{E(t)} \dot{E}(t) \right] + h_2 \left( \frac{B(t)}{E(t)} \right) \dot{q}_t. \] (60)
Especially when the vintage specific technological progress, $\hat{q}_t$, is constant, (60) is equivalent to

$$\frac{\dot{A}(t) - A(t)}{B(t) - B(t)} = \frac{\dot{b}(t)}{\beta b(t)} = h(b(t), \hat{q}_t) = \frac{\beta a(t)}{\alpha b(t)} \frac{ab(t) - (\beta - 1)a(t)}{\beta a(t) - (\alpha - 1)b(t)}$$

$$= \frac{\beta A(t) - (\beta - 1)A(t)}{\alpha B(t) - (\alpha - 1)B(t)}$$

(61)

4 Steady State

4.1 Characterization

When the growth rates of vintage specific technology, $\hat{q}_t$, is constant, which means $\tilde{\delta}_B(t) = \tilde{\delta}_B$, a sufficient condition of having a steady state $b^* > 0$ is

$$0 < \sigma < \frac{\tilde{\delta}_B}{\delta_B - \delta_A}$$

(62)

which is similar to the dis-embodied case, (15). Intuitively, this happens because if the depreciation of $A$ capital is higher than the sum of physical depreciation and obsolescence of $B$ capital, investment in old $A$ capital increases faster than production does, and total investment exceeds the production eventually.

Assuming (62) and constant growth of vintage non-specific technology, $\hat{z}(t)$ and labor, $L(t)$, full characterization of the steady state is useful in order to retrieve testable implications. The result is as follows:

**Proposition 5** Assume that the growth rates of labor, $\hat{L}$, vintage specific technology, $\hat{q}_t$, and vintage non-specific technology, $\hat{z}$, are constant and that parameters satisfy (62). Assume further that the economy is in the steady state. Then, we have a steady state $a^*$ and $b^*$ from (56) and given the current levels of $L(t)$, $q_t$, and $z(t)$, the economy is characterized as following:

1. Aggregate Capital

$$A(t) = a^*E(t)$$

$$B(t) = b^*E(t)$$

2. Change in Price and Net Depreciation Rate of Capital $B$

$$P^A(t) = P^B(t) = 1$$

$$\tilde{P}^B(t) = -\frac{\hat{q}}{\beta}$$

$$\delta_B = \delta^B - \tilde{P}^B(t) = \delta^B + \frac{\hat{q}}{\beta}$$

(63)
3. Distribution of labor

\[ L_v(t) = [\tilde{\delta}B + \bar{\dot{E}}] e^{-(\tilde{\delta}B + \bar{\dot{E}})(t-v)} L(t) \]

4. Distribution of capital

\[ A_v(t) = [\tilde{\delta}B + \bar{\dot{E}}] e^{-(\tilde{\delta}B + \bar{\dot{E}})(t-v)} A(t) \]

\[ B_v(t) = [\tilde{\delta}B + \bar{\dot{E}}] e^{-\delta B(t-v)} B(t) \]

5. Allocation of investment

\[ I^A_A(t) = [\tilde{\delta}B + \bar{\dot{E}}] A(t) \]

\[ I^A_A(t) = [\delta A - \tilde{\delta}B] A_v(t) \]

\[ I^A_A(t) = I^A_A(t) + \int_{-\infty}^{t} I^A_A(t) dv = [\delta A + \dot{E}] A(t) \quad (64) \]

\[ I^B_B(t) = I^B_B(t) = [\tilde{\delta}B + \bar{\dot{E}}] B(t) \quad (65) \]

where

\[ \bar{E} = \dot{E}(t) = \frac{\bar{z} + \bar{\dot{q}}}{1 - \alpha - \beta} + \bar{L}. \]

Proof. See Appendix.

The steady state result is somewhat similar to that of Aruga [1], although there is a difference in the investment pattern in vintage capital. In Aruga, investment in capital that works with old technology is only for the capital with higher depreciation rate. In the current model, investment in capital that works with old technology is only for the vintage non-specific capital.

Note that the investment in vintage \( A \) capital can be negative here because \( A \) capital can be freely reallocated. Indeed, the sign of the coefficient of (64) is negative when \( \delta A < \tilde{\delta}B \), which implies \( A \) capital working with old vintage \( B \) capital is disinvested and reallocated to newest capital. This happens because the depreciation rate of \( B \) capital is more than that of \( A \), and therefore there will be excess \( A \) capital in the vintage if there is no disinvestment. The disinvestment is proportional to the difference in the net depreciation rates, the sum of physical depreciation and obsolescence. Intuitively, following example can be explained by the model. At first, when firms construct a new factory, it builds building (\( A \)) and installs equipment (\( B \)) with newest technology. As time goes by, equipment in the factory depreciates faster than structure if \( \delta A < \tilde{\delta}B \) and the ratio \( A/B \) become larger than optimal, and thus reallocation of equipment occurs such that the existing building works with newest equipment by newer investment. Aruga’s model cannot explain this phenomena because both capital are vintage specific. When \( \tilde{\delta}A > \tilde{\delta}B \), new investment in \( A \) will be distributed to \( A \) that works with old vintage in order to complement the fast decreasing \( A \) compared to \( B \).
4.2 Comparative Statics

BEA publishes an extensive data about Fixed Assets. From this data, we can obtain stock of capital and investment of structure and equipment. As explained in the Introduction, one of the incentives to analyze the model is, relationships between the ratios of investment and stock of structure to equipment. General equilibrium models, such as Shell and Stiglitz [13] and Laitner and Stolyarov [12] shows they are proportional to the shares in production function:

\[
\frac{I^A(t)}{I^B(t)} = \frac{A(t)}{B(t)} = \frac{\alpha}{\beta},
\]

which is inconsistent with data. Here, we again the same result of Proposition 3 by replacing \(\delta^B\) with \(\tilde{\delta}^B(t)\).

We have the same result of Proposition 4 with respect to the neutral technological change. We have slightly different result with respect to vintage specific technological change:

**Proposition 6 (Change in Ratio of Stock and Investment when \(g\) changes)**

Assume the economy is in the steady state, which is characterized by the above proposition. Then, if the saving rate \(\sigma\) is below the Golden Rule, \(\sigma < \alpha + \beta\), a change in the rate of technological progress, \(g\), shifts the steady state equilibrium as following:

1. if \(\frac{\beta}{1-\alpha} \geq \frac{\delta^A + \tilde{\delta}^B}{\delta^A + \tilde{\delta}^B} \), then,

\[
\frac{d}{dq} \left[ \frac{A(t)}{B(t)} \right] \leq 0 \leq \frac{d}{dq} \left[ \frac{I^A(t)}{I^B(t)} \right],
\]

2. and otherwise,

\[
\frac{d}{dq} \left[ \frac{A(t)}{B(t)} \right] > 0 > \frac{d}{dq} \left[ \frac{I^A(t)}{I^B(t)} \right].
\]

**Proof.** See Appendix.

5 Empirical Analysis

5.1 Data

The data analyzed here is standard one. NIPA, JIP, SNA, Tangible Wealth, BLS (Gross Output), Gordon, and Cummins and Violante.

5.2 Ratio of Stock of Structure to Equipment and of Investment

We have seen the relationship of the ratio of the stock of structure to equipment and of investment in Figure 2. Figure 5 shows the same relationship in Japan.
In the both cases, we observe $I^A(t)/I^B(t) < A(t)/B(t)$, which is consistent with the Proposition 1 and 3 that are robust to embodiment assumption. This relationship cannot be obtained from other existing models, Shell and Stiglitz, and Laitner and Stolyarov. Their economy on the straight line $\alpha/\beta$ in Figure 4 as in (66) and therefore the ratio is inconsistent with Figure 2 and Figure 5.

By assuming the economy is in a steady state, we can estimate the value of $\alpha$ and $\beta$ using (72),

$$\beta x(w + 1) - \alpha(w + 1) + \sigma(w - x) = 0,$$

where $x = A(t)/B(t)$ and $w = I^A(t)/I^B(t)$. We can observe the value of $\alpha + \beta$ from the labor share of the data. Two unknowns, $\alpha$ and $\beta$, can be obtained from these two identities. Interestingly, as for the ratio $\alpha/\beta$, given the data of $x = A/B$ and $w = I^A/I^B$, we have

$$\frac{\alpha}{\beta} = \frac{(w + 1)x - [\sigma/(\alpha + \beta)](x - w)}{w + 1 + [\sigma/(\alpha + \beta)](x - w)},$$

which is a function of ratio of saving to capital share, $\sigma/(\alpha + \beta)$. Using this specification with data after 1960 that suggests $\alpha + \beta = 0.3$, we have a result $\alpha = \beta \approx 0.15$.\(^8\)

\(^8\)Note that embodied or disembodied does not affect the estimation.
Change of the ratio should be opposite as for Proposition 4 or 6. The data does not show this. Need to think about the possible reason. Again, bias in data? or transition dynamics?

Bias of price seems important. Bias in NIPA equipment price before 1973 is smaller than that after 1973 as for Gordon and Cummins and Violante. This affects the level of stock of capital. The upward bias of stock of equipment is more after 1973 then before 1973, which rotates the line $A(t)/B(t)$ in Figure 2 counter-clockwise.

Double counting of obsolescence might account for the inconsistency. If $g$ high, bias in $\delta B$ is high, and the stock of $B$ is under biased. If we believe $g$ goes down over time, the ratio $A/B$ shifts down more in older years.

Japan?

5.3 Productivity Growth and Change in Equipment Prices

Figure 6 shows labor productivity, multifactor productivity, and real price of equipment in the U.S. The number in year 2000 is normalized to 100%. As can be seen, the productivity monotonically grows and the real price of equipment monotonically decline over time, which is consistent with the equation (40).

Table 1 shows the growth rates of productivity and real price of equipment in the U.S., 1960-2002. Multifactor productivity and labor productivity are from
Table 1: Growth Rates of Multifactor Productivity, Labor Productivity, and Real Price of Equipment (%), 1960 - 2002.

<table>
<thead>
<tr>
<th>Year</th>
<th>MPFG</th>
<th>LPG</th>
<th>Adj LPG</th>
<th>NIP A</th>
<th>Gordon</th>
<th>C&amp;V</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960-2002</td>
<td>1.20</td>
<td>2.30</td>
<td>1.95</td>
<td>−2.33</td>
<td>−4.38</td>
<td>−4.45</td>
</tr>
<tr>
<td>1960-1974</td>
<td>1.87</td>
<td>2.95</td>
<td>2.76</td>
<td>−1.72</td>
<td>−4.08</td>
<td>−3.57</td>
</tr>
<tr>
<td>1975-2002</td>
<td>0.86</td>
<td>1.92</td>
<td>1.47</td>
<td>−2.96</td>
<td>−5.23</td>
<td>−5.34</td>
</tr>
</tbody>
</table>

BLS. Using BLS’s adjusted labor input index, adjusted labor productivity is also calculated because the raw number of the labor productivity from BLS does not care the change in labor quality. Prices are from NIPA, Gordon, Cummins and Violante. NIPA has some bias as documented in Gordon, and Cummins and Violante. The numbers under Gordon, and Cummins and Violante in the table is corrected growth rate of real prices.

The growth rate of vintage specific technological progress is, using (38),

\[ \hat{q} = -\beta \hat{P}_B(t). \]

The relationship of change in price in data and that of the model should be noted here. On the one hand, the real equipment prices in data shows how much, in the GDP prices at the day, was the prices of a unit of equipment that costs $1 today. For example, a unit of capital that costs $1 in 2000 cost $2.47 in 1960 in Figure 6. On the other hand, model’s prices (38) shows how much, in the GDP price today, is the price of a unit of capital that cost $1 at the day. In the example above, a unit of capital that cost $1 in 1960 costs only $0.405 today. Hence, we can obtain the change in prices that corresponds to (38) can be obtained by taking inverse of the price series of the data. [***Cite Jorgenson [11]???

The relationship of productivity growth and change in prices in a steady state has an accounting

\[ \text{MFP Growth} = (1 - \alpha - \beta) \ast \text{LP Growth} = \hat{E} = \hat{\epsilon} + \hat{q} = \hat{\epsilon} - \beta \hat{P}_B(t). \]

Using this accounting equation, we can estimate the importance of vintage specific change by using the value \( \beta = 0.15 \), which is derived above. The result is in Table 2. About 60% of the MFP growth is from vintage specific technological change as in GHK.

Can I add Japanese data?

Wedge between mpg and lpg????
Table 2: Decomposition of Productivity Change (%), 1960 - 2002.

<table>
<thead>
<tr>
<th>Year</th>
<th>Price Data</th>
<th>MFP, $\hat{q} + \hat{z}$</th>
<th>Specific $\hat{q}$</th>
<th>Neutral, $\hat{z}$</th>
<th>Contribution of $\hat{q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>NIPA</td>
<td>1.20</td>
<td>0.35</td>
<td>0.85</td>
<td>29%</td>
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<tr>
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6 Discussion

6.1 Difference from Existing Models


6.2 Investment Specific Technological Change and Neutral Productivity Growth

6.3 Bias in NIPA Capital Stock

Fraumeni. Double Counting of Obsolescence. Bias of NIPA price, Gordon, CV.

6.4 Difference in the Ratios of Capital Stock in the U.S. and Japan

7 Conclusion

Future study include; across country analysis using penn world table, ...

References


A Appendix

A.1 Proof of Proposition 1

(7) can be rewritten as
\[ a(t)^\alpha b(t)^\beta \left[ \frac{\beta}{b(t)} - \frac{\alpha}{a(t)} \right] = \delta^B - \delta^A > 0 \]
by assumption. Then, the inside of the parenthesis must be positive and therefore
\[ \alpha \beta < \frac{a(t)}{b(t)} = \frac{A(t)}{B(t)}. \]

■

A.2 Proof of Proposition 2

First, aggregate capital can be specified by
\[ \frac{A(t)}{E(t)} = a^*, \]
\[ \frac{B(t)}{E(t)} = b^*, \]
which are from the per effective labor definition.

Then, consider the investment allocation between aggregate intangible capital and aggregate tangible capital. Since at the steady state, \( \dot{a}(t) = \dot{k}(t) = 0 \) and \( \delta^B = \delta^B \), from (53) and (54) we have
\[ \frac{I^A(t)}{E(t)} = \left[ \delta^A + \hat{E} \right] a^*, \quad (67) \]
\[ \frac{I^B(t)}{E(t)} = \left[ \delta^B + \hat{E} \right] b^*, \quad (68) \]
where \( \hat{E} = \hat{E}(t) = \frac{\delta}{1-\alpha-\beta} + \hat{L}. \)

■

A.3 Proof of Proposition 3

We assume \( \delta^A < \delta^K \) without loss of generality. (47) and (55) imply that in the steady state, we have
\[ \beta(a^*)^\alpha (b^*)^{\beta-1} - a(a^*)^{\alpha-1} (b^*)^\beta = D - C > 0 \]
and
\[ \sigma(a^*)^\alpha (b^*)^\beta = Ca^* + Db^*, \]
where
\[ C = \delta^A + \hat{E}, \]
\[ D = \delta^B + \hat{E}. \]
By using the new notations,

\[ \theta = \frac{C}{D} (< 1 \text{ when } \delta^A < \delta^B), \]

\[ x = \frac{a^*}{b^*} (> \frac{\alpha}{\beta} \text{ when } \delta^A < \delta^B), \]  

(71)

(69) and (70) can be summarized as

\[ \beta \theta x^2 + [\beta - \alpha \theta - \sigma(1 - \theta)]x - \alpha = 0. \]  

(72)

Since \( \beta \theta > 0 \) and \(-\alpha < 0\), (72) has a unique positive root of \( x \), which is

\[ x = \frac{-(\beta - \alpha \theta - \sigma(1 - \theta)) + \sqrt{[\beta - \alpha \theta - \sigma(1 - \theta)]^2 + 4\alpha\beta\theta}}{2\beta\theta}. \]  

(73)

Now, check the change in \( x \) with the change in the growth rate, \( \dot{\zeta} \). The total differential of (72) gives

\[ \frac{dx}{d\theta} = \frac{[(\alpha - \sigma) - \beta x]x}{2\beta \theta x + [\beta - \alpha \theta - \sigma(1 - \theta)]}, \]  

(74)

which is negative.\footnote{Using the positive root of \( x \), (73), the sign of the denominator of (74) is \( 2\beta x + [\beta - \alpha \theta - \sigma(1 - \theta)] = \sqrt{[\beta - \alpha \theta - \sigma(1 - \theta)]^2 + 4\alpha\beta\theta} > 0 \). The numerator of (74) is negative because \( \alpha - \beta x < 0 \) from (71). Therefore, (74) is negative.}

Then, the sign of \( \frac{dx}{d\dot{\zeta}} \) depends on the sign of \( \frac{dx}{d\theta} \), which is clearly positive when \( \delta^A < \delta^B \). Therefore, we have

\[ \frac{dx}{d\dot{\zeta}} < 0 \]

We have similar analysis for the ratio of investment in \( A \) and \( B \). From (67) and (68), we can define the ratio of total investment in intangible capital and tangible capital as,

\[ w = \frac{I^A(t)}{I^B(t)} = \frac{C}{D} \frac{a^*}{b^*} = \theta x. \]

We are again interested in positive root, which is

\[ w = \theta x = \frac{-(\beta - \alpha \theta - \sigma(1 - \theta)) + \sqrt{[\beta - \alpha \theta - \sigma(1 - \theta)]^2 + 4\alpha\beta\theta}}{2\beta}. \]  

(75)

Note that

\[ w < \frac{\alpha}{\beta} \]

because value of the left hand side of (72) is \( \alpha(1 - \theta)(\alpha + \beta - \sigma)/\beta > 0 \) when \( w = \alpha/\beta \), and therefore we know \( w \in (0, \alpha/\beta) \).
The change in $w$ with the change in $\theta$ would be
\[
\frac{dw}{d\theta} = \frac{d(\theta x)}{d\theta} = \frac{\alpha + (\alpha - \sigma)w}{2\beta w + [\beta - \alpha\theta - \sigma(1 - \theta)]}.
\] (76)

(76) is positive when the saving rate is below the golden rule threshold, $\sigma < \alpha + \beta$.\(^{10}\)

Therefore investment rate, $w = I^A(t)/I^K(t)$, behaves opposite way against the ratio of aggregate capital, $x = a^*/k^*$ when golden rule holds. ■

A.4 Proof of Proposition 4

The first part of the analysis is the same as Proposition 1, except
\[
\hat{E} = \hat{z} + \hat{q} + \hat{L},
\]
and $\delta^B > \delta^B$.

Now, investment allocation between intangible capital and tangible capital is given by
\[
I^A(t) = A(t) + \int_{-\infty}^{t} I^A_v(t)dv = [\delta^A + \hat{E}]A(t),
\] (77)
\[
I^B(t) = B(t) = [\delta^B + \hat{E}]B(t).
\] (78)

(77) and (78) also imply the investment allocation between them is constant,
\[
\frac{I^B(t)}{I^A(t) + I^B(t)} = \frac{[\delta^B + \hat{E}](b^*/a^*)}{[\delta^A + \hat{E}] + [\delta^B + \hat{E}](b^*/a^*)} = \theta.
\]

Since the investment in $B$ capital is only for the frontier vintage, we have
\[
B_v(t) = B_v(\nu)e^{-\delta^B(t-\nu)} = \theta\sigma Y(\nu)e^{-\delta^B(t-\nu)}.
\] (79)

In a steady state, $\hat{\gamma}(t) = \alpha\hat{A}(t) + \beta\hat{k}(t) = 0$. This implies
\[
\hat{Y}(t) = \hat{E}(t) = \hat{E}
\]
and therefore
\[
Y(\nu) = Y(t)e^{-\hat{E}(t-\nu)}.
\]

Thus, (79) can be written as
\[
B_v(t) = \theta\sigma Y(t)e^{-(\delta^B + \hat{E})(t-\nu)}.
\] (80)

\(^{10}\)The sign of the denominator of (76) is positive as in the first case. From (75), the numerator of (76) is $\alpha + (\alpha - \sigma)w = w(\alpha/w + \alpha - \sigma) > w(\alpha + \beta - \sigma) > 0$ from the golden rule. Since both the denominator and the numerator are positive, (76) is positive.
Now, find the optimal allocation of labor, \( L_v(t) \). From (28), (38), and (80) with per effective labor notation, read

\[
L_v(t) = P_v(t) \frac{B_v(t)}{L_t(t)} = e^{-(\delta B + \hat{E})(t-v)} L_t(t).
\]  

(81)

So, since the total amount of labor is \( L(t) = \int_{-\infty}^{t} L_v(t)dv \), we have

\[
L_t(t) = [\hat{\delta} B + \hat{E}] L(t).
\]  

(82)

Therefore, (81) and (82) can determine the distribution of the labor, \( L_v(t) \).

(81) combined with (78) gives us distribution of capital,

\[
B_v(t) = L_v(t) L_t(t) = \frac{1}{[\hat{\delta} B + \hat{E}] L(t)} \left[ \tilde{\delta} B + \hat{E} \right] e^{-(\delta B + \hat{E})(t-v)} B(t).
\]  

(83)

Now consider \( A \) capital. Using (83), we have

\[
A_t(t) = B_t(t) \frac{A(t)}{B(t)} = \frac{\tilde{\delta} B + \hat{E}}{\delta A + \hat{E}} A(t).
\]

Therefore with (27),

\[
A_v(t) = A_t(t) \frac{L_v(t)}{L_t(t)} = \frac{\tilde{\delta} B + \hat{E}}{\delta A + \hat{E}} e^{-(\delta B + \hat{E})(t-v)} A(t).
\]

So, we have

\[
\hat{A}_v(t) = -\left( \tilde{\delta} B + \hat{E} \right) + \hat{A}(t) = -\delta B.
\]

Therefore,

\[
I_v^A(t) A_v(t) = \hat{A}_v(t) + \delta A = \delta A - \delta B,
\]

and thus

\[
I_v^A(t) = \left[ \delta A - \delta B \right] A_v(t).
\]

A.5 Proof of Proposition 5 and 6 [Incomplete]

The analysis is similar to that of Proposition 2 and 3. However, there is difference from the existence of vintage specific technology, which affects the behavior of

\[
\theta = \frac{\delta A + \frac{\hat{\delta} B + \hat{L}}{1-\alpha-\beta} + \tilde{L}}{\delta B + \frac{\hat{q}}{\beta} + \frac{\hat{\delta} + \hat{L}}{1-\alpha-\beta} + \tilde{L}}.
\]

The analysis with respect to \( \hat{\delta} \) is the same as proposition 2.

As for \( \hat{q} \),

\[
\frac{d\theta}{dq} = \frac{\beta(\delta B + \tilde{L}) - (1-\alpha)(\delta A + \tilde{L})}{\beta(1-\alpha-\beta)[\delta B + \hat{E}]^2},
\]  

(84)
which can be either positive or negative. The sign of denominator of (84) is positive, and numerator can be either positive or negative. The sign of numerator is related to the range of $\theta$. It’s easy to see that $\theta \to \beta/(1 - \alpha) < 1$ as $\hat{q} \to \infty$, and $\theta = (\delta^A + \hat{L})/(\delta^B + \hat{L}) < 1$ when $\hat{q} = 0$. So, if
\[
\frac{\beta}{1 - \alpha} \geq \frac{\delta^A + \hat{L}}{\delta^B + \hat{L}},
\]
then
\[
\frac{d\theta}{dq} \geq 0,
\]
and else
\[
\frac{d\theta}{dq} < 0.
\]