493 ADDITIONAL EXERCISE SET 2

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1. The Second Isomorphism Theorem

Prove the following.

(1) *Normality Transfer Condition.* If $H, N \subseteq G$ are subgroups and N is normal in H, the $H \cap N$ is normal in H.
(2) *Intermediate Subgroup Condition.* If $H \subseteq K \subseteq G$ are groups and H is normal in G, then H is normal in K.
(3) Let G be a group. Let S be a subgroup of G, and let N be a normal subgroup of G. Prove the following:
  (a) The product SN is a subgroup of G.
  (b) The intersection $S \cap N$ is a normal subgroup of S.
  (c) The quotient groups $(SN)/N$ and $S/(S \cap N)$ are isomorphic.
  Exercise 3c is called the Second Isomorphism Theorem.

2. Product Groups

Prove the following.

(1) *Universal Property of Group Products.* Let G, H be groups, and let $\pi_G : G \times H \to G$, $\pi_H : G \times H \to H$ be the projection homomorphisms.
  (a) Prove that for any group P and any homomorphisms $f_G : P \to G$, $f_H : P \to H$, there exists a unique homomorphism $f : P \to G \times H$ such that
  $$\pi_G \circ f = f_G, \ \pi_H \circ f = f_H.$$
  (b) Prove $G \times H$ is the unique group with the property described above.
(2) Let G and H be groups, and let $\alpha : G \to G, \beta : H \to H$ be automorphisms.
  (a) Show that the product function $\alpha \times \beta : G \times H \to G \times G$ defined by
  $$\alpha \times \beta : (g, h) \mapsto (\alpha g, \beta h)$$
  is an automorphism of $G \times H$.
  (b) Conclude that the function
  $$\Psi : \mathrm{Aut}(G) \times \mathrm{Aut}(G) \to \mathrm{Aut}(G \times H)$$
  is an injective group homomorphism.
(3) A general element of $\mathrm{Aut}(G \times H)$ is of the form
  $$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$
  where $\alpha \in \mathrm{Aut}(G)$, $\delta \in \mathrm{Aut}(H)$, $\beta \in \mathrm{Hom}(H, G)$ and $\gamma \in \mathrm{Hom}(G, H)$. Write down the action of such matrices on $G \times H$, and classify those which induce automorphisms.

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