493 ADDITIONAL EXERCISE SET 1

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1. Review of Quotients

In this section, do all the problems and be prepared to discuss at least the ones with a $\star$.

(1) Let G be a finite group and $H \subseteq G$ be a subgroup of G
   (a) Show that $g_1H = g_2H$ iff $g_1^{-1}g_2 \in H$
   (b) Show that the left cosets $\{gH : g \in G\}$ of H partition G; i.e., the relation $\sim$ defined by $g_1 \sim g_2$ provided that $g_1H = g_2H$ is an equivalence relation.
   (c) Show that for any $g_1, g_2 \in G$ there is a bijection $g_1H \to g_2H$, and hence all left cosets of H in G have the same cardinality.
   $\star$ (d) The number of left cosets of H in G is called the *index* of H in G, denoted $[G : H]$. Prove *Lagrange's Theorem*: $|G| = |H| \cdot [G : H]$.
   $\star$ (e) Explain why the order of a subgroup of a finite group must divide the order of the group (this is also called Lagrange's Theorem).

(2) Show that $H \trianglelefteq G$ if and only if the left cosets of H are the same as the right cosets, i.e. $gH = Hg$ for all $g \in G$.

(3) Let G/H denote the set $\{gH : g \in G\}$ of left cosets of H in G, and let $H\ G$ denote the set $\{Hg : g \in G\}$ of right cosets .
   (a) Show that $H \trianglelefteq G$ if and only if $gH = Hg$ for all $g \in G$.
   $\star$ (b) Show that the binary operation

$$\begin{aligned}(G/H) \times (G/H) &\to G/H \\ (gH, hH) &\mapsto (gh)H\end{aligned}$$

   is well defined if and only if $H \trianglelefteq G$. [Hint: use (1a).]
   $\star$ (c) Show that in the case $H \trianglelefteq G$, the map

$$\begin{aligned}\pi\ :\ G &\to G/H \\ g &\mapsto gH\end{aligned}$$

   is a group homomorphism, and hence G/H is a group. (Wait, what?)
   (d) Construct an isomorphism $\phi : H \to \ker(\pi)$.

(4) Let G be a group. For $x, y \in G$, define $[x, y] = xyx^{-1}y^{-1}$. Now let

$$[G, G] = \{[x, y] \mid x, y \in G\}.$$

   (a) Prove that $[G, G] \trianglelefteq G$.
   (b) Prove that $G/[G, G]$ is abelian.
   $\star$ (c) Prove that if H is an abelian group and $\phi : G \to H$ is a group homomorphism, then there exists a unique homomorphism $\bar{\phi} : G/[G, G] \to H$ such that

$$\bar{\phi} \circ \pi = \phi.$$

Date: September 30, 2018.

2. The First Isomorphism Theorem

Be prepared to discuss your solution to *all* of the following problems.

(1) Let G and H be groups, and let $\phi : G \to H$ be a homomorphism. Show that
  (a) $\ker(\phi) \trianglelefteq G$,
  (b) $\mathrm{im}(\phi) \subseteq H$ (here "$\subseteq$" means "subgroup"),
  (c) $\mathrm{im}(\phi) \cong G/\ker(\phi)$,
  (d) If ϕ is surjective, then $H \cong G/\ker(\phi)$.

Exercise (1c) is called the First Isomorphism Theorem.

(2) Let G be a finite group of order 21 and let K be a finite group of order 49. Suppose that G does not have a normal subgroup of order 3. Then determine all group homomorphisms from G to K.

(3) Let G be a finite group and let N be a normal abelian subgroup of G. Let $\mathrm{Aut}(N)$ be the group of automorphisms of N. Suppose that the orders of groups G/N and $\mathrm{Aut}(N)$ are relatively prime. Then prove that N is contained in the center of G.