396 DISCUSSION SECTION 3

WITH NOAH LUNTZLARA

Whenever we have a group action $G \circlearrowright S$, we get a *quotient set*, S/G, which simply denotes the set of G-orbits in S.

(Ex. 1) Hello! Today we're going to prove the Cauchy-Frobenius Lemma in a few steps. Let $G \circlearrowright S$. Recall that $\text{fix}(g)$ denotes the set $\{s \in S \,|\, gs = s\}$ of points that are fixed by g.

(a) Show that $\sum_{g \in G} \text{fix}(g) = |\{(g, s) \in G \times S : g \cdot s = s\}|$.

(b) Show that $|\{(g, s) \in G \times S : g \cdot s = s\}| = \sum_{s \in S} |\text{Stab}(s)|$.

(c) Show that $\sum_{s \in S} |\text{Stab}(s)| = |G| \cdot |S/G|$

(d) Conclude that $|G| \cdot |S/G| = \sum_{g \in G} |\text{fix}(g)|$

(Ex. 2) Use Burnside's Lemma to count the number of non-isomorphic graphs on four vertices. Check your answer by drawing all of them.

(Ex. 3) Use Burnside's Lemma to count the number of ways to 2-color the sides of an octahedron (my favorite solid).

(Ex. 4) Here are the Sylow theorems, according to Wikipedia. Can you prove them yet?

(a) Theorem 1: For every prime factor p with multiplicity n of the order of a finite group G, there exists a Sylow p-subgroup of G, of order pn.
The following weaker version of theorem 1 was first proved by Cauchy, and is known as Cauchy's theorem.
Corollary: Given a finite group G and a prime number p dividing the order of G, then there exists an element (and hence a subgroup) of order p in G.

(b) Theorem 2: Given a finite group G and a prime number p, all Sylow p-subgroups of G are conjugate to each other, i.e. if H and K are Sylow p-subgroups of G, then there exists an element g in G with g1Hg = K.

(c) Theorem 3: Let p be a prime factor with multiplicity n of the order of a finite group G, so that the order of G can be written as pnm, where n ¿ 0 and p does not divide m. Let np be the number of Sylow p-subgroups of G. Then the following hold:
np divides m, which is the index of the Sylow p-subgroup in G. np 1 (mod p). np = —G : NG(P)—, where P is any Sylow p-subgroup of G and NG denotes the normalizer.

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