395 DISCUSSION SECTION 7 OR SOMETHING

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Recall the definition of a group action: given a group G and a set S, an *action of G on S* is a map

$$\begin{aligned} \cdot : G \times S &\to S \\ (g, s) &\mapsto g \cdot s \end{aligned}$$

which satisfies the properties

- $e \cdot s = s$ for all $s \in S$
- $g \cdot (h \cdot s) = (gh) \cdot s$ for all $g, h \in G, s \in S$.

(We often suppress the "$\cdot$" where unambiguous.)

Suppose for the rest of this worksheet that G is a group, S is a set, and G acts on S.

(Ex. 1) Fix $g \in G$. Show that the induced map

$$\begin{aligned} \sigma_g : S &\to S \\ s &\mapsto gs \end{aligned}$$

is a bijection.

(Ex. 2) Give four examples of G acting on itself.

(Ex. 3) (a) Prove *Cayley's theorem*: every finite group G is a subgroup of the symmetric group S_n for some $n \in \mathbb{N}$.
(b) Every group acts on the singleton set $\{s\}$. Can your proof in the previous part be strengthened to show every group is a subgroup of the trivial group? If so, fix it.

The *orbit* of $s \in S$ is the set

$$\mathrm{Orb}_G(s) = \{gs \mid g \in G\}.$$

Notice that this is a **<u>subset of S</u>**.

(Ex. 4) (a) Show that the orbits partition S.
(b) Show that when G is finite, the orbits have order dividing $|G|$.

(Ex. 5) Let G act on itself by *conjugation*: $g \cdot h = ghg^{-1}$. The orbits under this action are called *conjugacy classes*.
(a) The union of all conjugacy classes with order 1 is called the *center* of G, denoted $Z(G)$. Show that $g \in Z(G)$ if and only if it commutes with every element of G.
(b) Since the orbits partition G, we can write the order of G as the sum of the orders of its conjugacy classes:

$$|G| = \sum_{\mathcal{O} \in \{\mathrm{Orb}_G(s) \,|\, s \in S\}} |\mathcal{O}|.$$

This is called the *class equation* for G. Write down the class equation for
(i) $G = \mathbb{Z}/3\mathbb{Z}$
(ii) $G = S_3$
(iii) $G = D_4$

The *stabilizer* of $s \in S$ is the set

$$\mathrm{Stab}_G(s) = \{g \in G \mid gs = s\}.$$

Note that this is a **<u>subset of G</u>**.

(Ex. 6) [D. Copeland] Show that the stabilizer of n in the action of S_n on $\{1, \ldots, n\}$ is isomorphic to S_{n-1}.

(Ex. 7) Show that $\mathrm{Stab}_G(s)$ is a normal subgroup of G.

(Ex. 8) [D. Copeland] Find the orbits and stabilizers for the following parts of a cube. Notice anything?
(a) a single vertex
(b) an edge
(c) a face

(Ex. 9) State and prove the *Orbit-Stabilizer Theorem*.

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