395 DISCUSSION SECTION 2

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We start with the problems we didn't get to last time:

(Ex. 5) Let G be a group. Show that a nonempty set $H \subseteq G$ is a subgroup iff $a, b \in H \Rightarrow ab^{-1} \in H$.

(Ex. 6) What is the order of a group G generated by two elements x and y subject only to the relations

$$x^3 = y^2 = (xy)^2 = e?$$

List all the subgroups of G.

The order of a group is simply the size of the set G. The order of an element $g \in G$ is the smallest $i \in \mathbb{N}$ such that $g^i = 0$

(Ex. 7) Show that every finite group with an even order has an element of order 2.

Recall the definition of *nth symmetric group*: $S_n := \{\text{bijections } \mathbb{N}_n \to \mathbb{N}_n\}$ (where $\mathbb{N}_n$ denotes the set $\{1, 2, \ldots, n\}$). Also recall the cycle notation we use to denote elements of S_n; for example, $(125)(46) \in S_6$ is the map which takes

$$1 \mapsto 2 \mapsto 5 \mapsto 1, \; 3 \mapsto 3, \; 4 \mapsto 6 \mapsto 4.$$

(Ex. 8) Prove that the cycle decomposition in S_n is unique.

(Ex. 9) Prove that the order of a permutation $\sigma \in S_n$ is the least common multiple of the cycle lengths.

(Ex. 10) *Bonus:* What is the highest order element of S_n in terms of n?

Let G be a group, and $S \subseteq G$ be an arbitrary subset of G. Recall the definitions of the (*left* and *right*) *cosets* of S in G: for each $g \in G$, we define

$$gS := \{gx \mid x \in S\}; \qquad \text{respectively,} \qquad Sg := \{xg \mid x \in S\}.$$

(Ex. 1) When are right cosets and left cosets the same? When are they different? Give some examples.

(Ex. 2) Let G be a finite group and $H \subseteq G$ be a subgroup of G

- (a) Show that $g_1H = g_2H$ iff $g_1g_2^{-1} \in H$
- (b) Show that the left cosets of H partition G; i.e., the relation $\sim$ defined by $g_1 \sim g_2$ provided that $g_1H = g_2H$ is an equivalence relation.
- (c) Show that for any $g_1, g_2 \in G$ there is a bijection $g_1H \to g_2H$, and hence all left cosets of H in G have the same cardinality.
- (d) The number of left cosets of H in G is called the *index* of H in G, denoted $[G : H]$. Prove *Lagrange's Theorem*: $|G| = |H| \cdot [G : H]$.
- (e) Explain why the order of a subgroup of a finite group must divide the order of the group (this is also called Lagrange's Theorem).

(Ex. 3) Show that when we do everything above with right cosets instead of left cosets, we get the same number for $[G : H]$.

(Ex. 4) Classify all groups with no proper nontrivial subgroups (i.e., subgroups that are neither the group itself nor the trivial subgroup $\{e\}$).

A subgroup H of a group G is called *normal* provided that $gHg^{-1} = H$ for every $g \in G$ (where $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$). If H is a normal subgroup of G, we write $H \trianglelefteq G$.

(Ex. 5) Show that $H \trianglelefteq G$ iff $gH = Hg$ for every $g \in G$.

(Ex. 6)
- (a) Show that for every group G, $G \trianglelefteq G$ and $\{e\} \trianglelefteq G$.
- (b) Exhibit three groups with non-normal subgroups.
- (c) Exhibit three groups with proper nontrivial normal subgroups.

(Ex. 7) Show that if $H \trianglelefteq G$ and K is a subgroup of G which contains H, then $H \trianglelefteq K$.

(Ex. 8) *Bonus:* Try and fail to prove $H \trianglelefteq K \trianglelefteq G \Rightarrow H \trianglelefteq G$. Then find a group G with subgroups H and K such that $K \trianglelefteq G$ and $H \trianglelefteq K$, but H is not normal in G.

(Ex. 9) Let G be a group and H be an index-2 subgroup, so that $[G : H] = 2$. Show that $H \trianglelefteq G$.

(Ex. 10) Recall that if $\phi : G \to H$ is a group homomorphism, $\ker(\phi)$ denotes $\phi^{-1}(e_H)$.
- (a) Show that $\ker(\phi) \trianglelefteq G$.
- (b) State and prove a converse for the previous statement.

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