395 DISCUSSION SECTION 1

WITH NOAH LUNTZLARA

Recall that a *group* is an ordered pair $(G, *)$ consisiting of a set G and an associative binary operator $* : G \times G \to G$ with identity and inverses. We often use G to refer to the group, rather than the set. Where unambiguous, we also write ab for the image of the ordered pair (a, b) under $*$.

(Ex. 1) Give ten examples of groups, each more interesting than the previous.

(Ex. 2) Show that the *cancellation law* $ab = ac \Rightarrow b = c$ is equivalent to the existence of inverses.

A group is called *abelian* provided that the binary operator is commutative; i.e., $ab = ba$ for all $a, b, \in G$.

(Ex. 3) Let G be a group where every element $g \in G$ satisfies $g^2 = e$. Prove that G is abelian.

(Ex. 4) Find (with justification) the smallest non-abelian group.

A *subgroup* of a group G is a subset $H \subseteq G$ which forms a group with respect to the inherited binary operation.

(Ex. 5) Let G be a group. Show that a nonempty set $H \subseteq G$ is a subgroup iff $a, b \in H \Rightarrow ab^{-1} \in H$.

(Ex. 6) What is the order of a group G generated by two elements x and y subject only to the relations

$$x^3 = y^2 = (xy)^2 = e?$$

List all the subgroups of G.

The order of a group is simply the size of the set G. The order of an element $g \in G$ is the smallest $i \in \mathbb{N}$ such that $g^i = 0$

(Ex. 7) Show that every finite group with an even order has an element of order 2.

Recall the definition of *nth symmetric group*: $S_n := \{\text{bijections } \mathbb{N}_n \to \mathbb{N}_n\}$ (where $\mathbb{N}_n$ denotes the set $\{1, 2, \ldots, n\}$). Also recall the cycle notation we use to denote elements of S_n; for example, $(125)(46) \in S_6$ is the map which takes

$$1 \mapsto 2 \mapsto 5 \mapsto 1, \ 3 \mapsto 3, \ 4 \mapsto 6 \mapsto 4.$$

(Ex. 8) Prove that the cycle decomposition in S_n is unique.

(Ex. 9) Prove that the order of a permutation $\sigma \in S_n$ is the least common multiple of the cycle lengths.

(Ex. 10) *Bonus:* What is the highest order element of S_n in terms of n?

Date: September 15, 2017.