Symmetry (Part 2)

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NOTE: Most of the terminology on this sheet is non-standard. I made it up to help you think about symmetry in a particular way.

Definition. *A* permutation *of a set A is a bijection $A \to A$.*

Definition. *Let $f\colon A \to A$. A subset $S \subseteq A$ is said to be f-*invariant *provided that $f(S) \subseteq S$.*

1 More Invariants

Recall. An *invariant* is something which does not change when certain maps are applied. It is a very broad term and can refer to lots of different things. Finding an invariant can often be useful in solving a problem.

An f-invariant subset (defined above) is an example of an invariant. Let's give twenty more examples! Each example contains three pieces of information: a set A; a description of the invariant; what it means for a permutation $f\colon A \to A$ to preserve the invariant. So for example, the first row is the definition of an invariant set, above.

A	**invariant**	**f preserves the invariant**
any set	*any subset* $S \subseteq A$	$f(S) \subseteq S.$
any set	*any subset* $S \subseteq A$	$f(S) = S$
$\mathbb{R}$	$d\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}_{\geq 0}\colon (x,y) \mapsto \lvert x-y\rvert$	$\forall\, x,y \in \mathbb{R},\ d(f(x),f(y)) = d(x,y)$
$\mathbb{R}$	$m\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}\colon (x,y) \mapsto x-y$	$\forall\, x,y \in \mathbb{R},\ m(f(x),f(y)) = m(x,y)$
$\mathbb{R}^4$	$S:\mathbb{R}^4\times\mathbb{R}^4 \to \mathbb{R}\colon ((t_1,x_1,y_1,z_1),(t_2,x_2,y_2,z_2)) \mapsto$ $c^2(t_2-t_1)^2-(x_2-x_1)^2-(y_2-y_1)^2-(z_2-z_1)^2,$ *where c is the speed of light*	$\forall\, a,b \in \mathbb{R}^4, S(f(a),f(b)) = S(a,b)$
any set	$d\colon A\times A \to \mathbb{R}_{\geq 0}$ *a metric*	$\forall\, x,y \in A,\ d(f(x),f(y)) = d(x,y)$
any set	$R \subseteq A\times A$ *an equiv. relation*	$sRt \implies f(s)Rf(t)$
any set	$R \subseteq A\times A$ *an equiv. relation*	$sRt \iff f(s)Rf(t)$
$\mathbb{Z}$	$\equiv_n \subseteq \mathbb{Z}\times\mathbb{Z}, a \equiv_n b$ *p.t. n divides $(a-b)$*	$a \equiv_n b \implies f(a) \equiv_n f(b)$
$\mathbb{Z}$	$\equiv_n \subseteq \mathbb{Z}\times\mathbb{Z}, a \equiv_n b$ *p.t. n divides $(a-b)$*	$a \equiv_n b \iff f(a) \equiv_n f(b)$
$\mathbb{Q}$	$+\colon \mathbb{Q}\times\mathbb{Q} \to \mathbb{Q}\colon (a,b) \mapsto a+b$	$\forall a,b \in \mathbb{Q}, f(a+b) = f(a)+f(b)$
$\mathbb{Q}$	$\cdot\colon \mathbb{N}\times\mathbb{Q} \to \mathbb{Q}\colon (n,q) \mapsto nq$	$\forall n \in \mathbb{N}, q \in \mathbb{Q}, f(n\cdot x) = n\cdot f(x)$
$\mathbb{Q}$	$F\colon \mathbb{Q} \to \mathbb{Q}\colon m/n \mapsto 1/n$	$F \circ f = F$
$\mathbb{R}$	$C = \{f\colon \mathbb{R}\to\mathbb{R} \mid f$ *continuous*$\}$	$g \in C \implies f \circ g \in C$
$\mathbb{R}$	$\{I \subseteq \mathbb{R} \mid I$ *is an interval*$\}$	I *an interval* $\implies f(I)$ *an interval*
$\mathbb{R}^2$	$(a, \{(r,s) \mid r,s$ *rays from* $a\}), a \in \mathbb{R}^2$	$\forall\, r,s$ *rays from* $a,\ \angle(f(r),f(s)) = \angle(r,s)$
t.s. X	*the set of open sets* τ_X	$f(U) \in \tau_X \iff U \in \tau_X$
ordered field F	*the positive set P*	$a > b \implies f(a) > f(b)$
any set	$\{A_i\}_{i\in\mathcal{I}}$ *s.t.* $A = \cup_{i\in\mathcal{I}} A_i, A_i \cap A_j = \emptyset$	$\forall\, i \in \mathcal{I}, \exists j \in \mathcal{I}$ *s.t.* $f(A_i) \subseteq A_j$
any set	*a collection of invariants*	*for each invariant, f preserves it*

Definition. *Two invariants $\mathfrak{I}, \mathfrak{K}$ on a set A are said to be* equivalent *p.t. f preserves $\mathfrak{I} \iff f$ preserves $\mathfrak{K}$.*

(Ex. 1) (a) Prove that the invariants in the 9th and 10th rows of the table are not equivalent.
(b) Prove that the invariants in the 11th and 12th rows of the table are equivalent.

Definition. *An invariant $\mathfrak{I}$ is called* symmetric *provided that the following hold:*

I. $Id_A : A \to A$ given by $Id_A(x) = x$ preserves $\mathfrak{I}$.

II. If f, g are permutations of A and both f and g preserve $\mathfrak{I}$, then $f \circ g$ preserves $\mathfrak{I}$.

III. If f is a permutation of A that preserves $\mathfrak{I}$, then f^{-1} preserves $\mathfrak{I}$.

(Ex. 2) Which of the invariants in the table are symmetric?

Definition. *Let A be a set with a symmetric invariant $\mathfrak{I}$. The set of all permutations $f \colon A \to A$ which preserve $\mathfrak{I}$ is called the* symmetries *of A under $\mathfrak{I}$, denoted* $\mathrm{Group}(A, \mathfrak{I})$.

(Ex. 3) Find $\mathrm{Group}(A, \mathfrak{I})$ for five of the (set,invariant) pairs that you showed were symmetric in (Ex. 2).

(Ex. 4) Let R be a relation on a set A, and let $\mathfrak{I}_R$ be the invariant described on the eight row of the table. Let $\{A_i\}$ be the set of equivalence classes, and let $\mathfrak{K}_R$ be the invariant described on the 19th row of the table.

(a) Show that $\mathfrak{I}_R$ and $\mathfrak{K}_R$ are symmetric.
(b) Show that $\mathfrak{I}_R$ and $\mathfrak{K}_R$ are equivalent.

(Ex. 5) How many inequivalent symmetric invariants are there on A when

(a) $A = \emptyset$?
(b) $A = \mathbb{N}_1$?
(c) $A = \mathbb{N}_2$?
(d) $A = \mathbb{N}_3$?
(e) $A = \mathbb{N}_4$?