Symmetry (Part 1)

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1 Permutations

(1) How many ways are there to order the letters `M,A,T,H`?

(2) How many ways are there to order the letters `M,I,S,S,I,S,S,I,P,P,I`?

Definition. *A* permutation *of a set A is a bijection $A \to A$.*

(3) Prove that the number of permutations of $\mathbb{N}_n$ is $n!$.

(4) Prove that if there is a bijection $A \to B$, then there is a bijection between the set of permutations of A and the set of permutations of B. (A and B are not necessarily finite!)

2 Invariants

An *invariant* is something which does not change when certain maps are applied. It is a very broad term and can refer to lots of different things. Finding an invariant can often be useful in solving a problem.

(5) Beatrice writes the numbers 1, 2, 3, 4, 5, and 6 on a blackboard. Benedick selects two of these numbers, erases both of them, and writes down their sum on the blackboard. Benedick continues until there is only one number left on the board. What are the possible values of that number?

(6) Prove that you can't tile a chessboard with two opposite corner squares removed with 1×2 and 2×1 dominos.

(7) On the island of Camelot live 13 gray, 15 brown and 17 crimson chameleons. If two chameleons of different colors meet, they both simultaneously change color to the third color (e.g. if a gray and a brown chameleon meet each other, they both change to crimson).

  (a) Is it possible that they will eventually all be the same color?

  (b) Is it possible that there will eventually be the same numbers of gray, brown, and crimson chameleons?

Definition. *Let $f\colon A \to A$. A subset $S \subseteq A$ is said to be f-*invariant *provided that $f(S) \subseteq S$.*

(8) Prove that for any set A and any function $f\colon A \to A$, the sets A and $\emptyset$ are f-invariant.

(9) Let $a \in \mathbb{R}$. Define $t_a\colon \mathbb{R} \to \mathbb{R}$ by $t_a(x) = x + a$. Find all t_a-invariant intervals.[1] [Hint: break into cases $a = 0$, $a \neq 0$.]

(10) Let $a \in \mathbb{R}$. Define $m_a\colon \mathbb{R} \to \mathbb{R}$ by $m_a(x) = ax$. Find all m_a-invariant intervals.[2] [Hint: break into cases $a > 1$, $a = 1$, $0 < a < 1$, $a = 0$, $-1 < a < 0$, $a = -1$, $a < -1$.]

[1] The sheet originally said to find all t_a-invariant *sets*, but we realized this is too broad. Can you find some examples of sets that might have convinced us of this?

[2] The sheet originally said to find all m_a-invariant *sets*, but we realized this is too broad. Can you find some examples of sets that might have convinced us of this?