Cardinality and Topological Spaces

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1 Finite, infinite, and uncountable sets

A set A is called *countable* if there is a bijection between A and a subset of the natural numbers $\mathbb{N} := \{1, 2, 3, \dots\}$.

Question. *Prove the following.*
(i) A is countable if and only if $A = \emptyset$ or there is a surjective function $f : \mathbb{N} \to A$.
(ii) If A is infinite and countable then there is a bijection between A and $\mathbb{N}$.

Theorem. *Every infinite subset of a countable set A is countable.*

Union and intersection of sets

Question. *Prove the following:*

(i) $B \setminus A = A^c \setminus B^c$.

(ii) $(\bigcup_\alpha B_\alpha) \setminus (\bigcup_\alpha A_\alpha) \subset \bigcup_\alpha (B_\alpha \setminus A_\alpha)$.

Theorem. *Let $\mathcal{E} = \{E_n\}_{n\in\mathbb{N}}$ be a collection of countable sets indexed by $\mathbb{N}$, and put*

$$S = \bigcup_{n=1}^{\infty} E_n$$

Then S is countable.

Corollary. *Suppose A is a countable set, and, for every $\alpha \in A$, B_α is a countable set. Put*

$$T = \bigcup_{\alpha\in A} B_\alpha$$

Then T is countable.

Theorem. *If $A_1, \dots, A_k$ are countable sets then the Cartesian product $A_1 \times \cdots \times A_k$ is countable.*

Theorem. *The set of all rational numbers is countable.*

Note: In fact, even the set of all algebraic numbers is countable.

Theorem. *Let A be the set of all functions $f : \mathbb{N} \to \{0, 1\}$. This set A is uncountable.*

Metric Spaces

Open and closed sets in metric spaces

Question. *Is a set necessarily either open or closed? Can a set be both? Can it be neither?*

Topological Spaces

1) Topology

Question. *Let $W = \{B, H, P\}$ and set $\tau_W = \{\emptyset, W, \{H\}, \{B, P\}\}$. Verify that (W, τ_W) is a topological space.*

Question. *Let $\tau_R := \{(a, \infty)|a \in \mathbb{R})\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$. Show that $(\mathbb{R}, \tau_R)$is a topological space.*

2) Open and closed sets in a topological space

3) Basis and sub-basis

Remark. *If $\mathcal{B}$, as a basis of X, is countable, we say that X is **second counatble**.*

Question. *Show that the set of rational balls $\mathcal{B} = \{B(r, q)|r \in \mathbb{Q}^n, q \in \mathbb{Q}\}$ is a basis for $\mathbb{R}^n$ (with Euclidean Topology).*

4) Hausdorff spaces

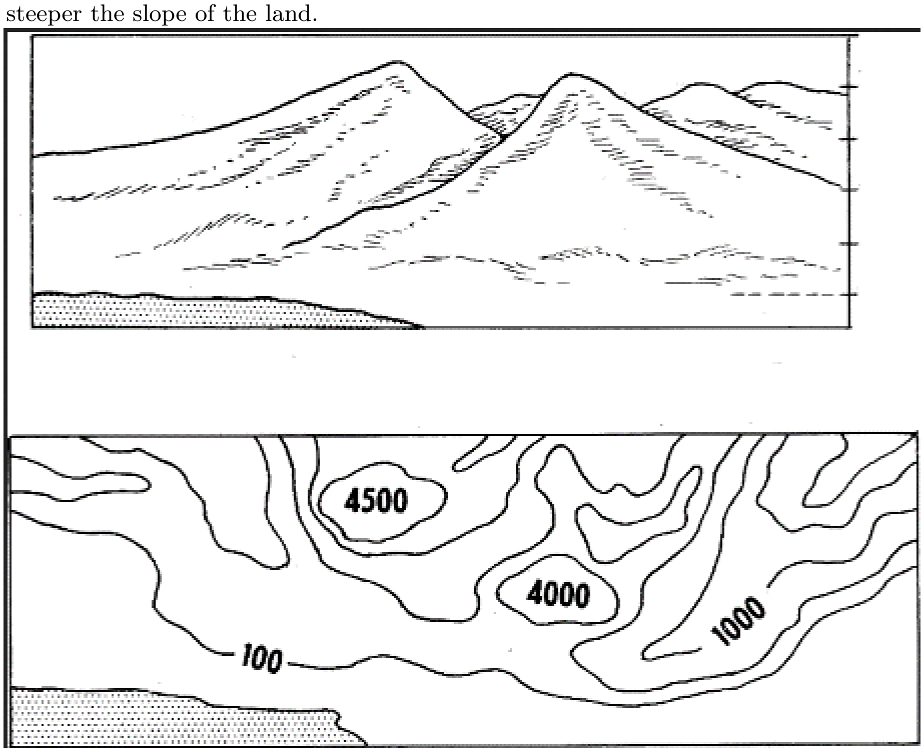
Let $(X, \mathcal{T})$ be a topological space, we say X is *Hausdorff* if $\forall x, y \in X$ with $x \neq y$, there are disjoint open sets $U_x, V_y \in \mathcal{T}$ such that $x \in U_x$ and $y \in V_y$.
Note: A metric space is automatically Hausdorff (Why?)

Theorem. *X is Hausdorff if and only if $\Delta = \{(x, x)|x \in X\} \subset X \times X$ is closed in $X \times X$ in the product topology. (We'll prove it after introducing product topology)*

A topographical map is one that shows the physical features of the land. Besides just showing landforms such as mountains and rivers, the map also shows the elevation changes of the land. Elevation is shown using contour lines.

When a contour line is drawn on a map it represents a given elevation. Every point on the map touching the line should be the same elevation. On some maps, numbers on the lines will let you know what the elevation is for that line.

Contour lines next to each other will represent different elevations. The closer the contour lines are to each other, the steeper the slope of the land.



Contour map example The lower map shows the contour lines for the above hills