Sets, Functions, and Relations

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1 Sets and Functions

Suppose X and Y are sets. Let $f : X \to Y$ be a function. Suppose $A \subset X$ and $B \subset Y$.

(1) Show $A \subset f^{-1}[f[A]]$.

(2) Is $f^{-1}[f[A]]$ always equal to A? If yes, prove it. If not, give a condition for which functions $f : X \to Y$ it is true that $A = f^{-1}[f[A]]$ for all $A \subset X$.

(3) Show $f[f^{-1}[B]] \subset B$.

(4) Is $f[f^{-1}[B]]$ always equal to B? If yes, prove it. If not, give a condition for which functions $f : X \to Y$ it is true that $B = f[f^{-1}[B]]$ for all $B \subset Y$.

(5) Suppose X and Y are sets. Let $f : X \to Y$ be a function. Suppose $A, C \subset X$ and $B, D \subset Y$. Fill in the blank with $\subset$, $\supset$, or $=$.

  (a) $f^{-1}[B \cap D] \qquad f^{-1}[B] \cap f^{-1}[D]$
  (b) $f[A \cap C] \qquad f[A] \cap f[C]$
  (c) $f[A \cup C] \qquad f[A] \cup f[C]$
  (d) $f^{-1}[B \cup D] \qquad f^{-1}[B] \cup f^{-1}[D]$
  (e) $f[A \setminus C] \qquad f[A] \setminus f[C]$

2 Equivalence relations

For each of the following relations, determine whether or not it is an equivalence relation, and

I. If it is not an equivalence relation, which equivalence relation property it fails to satisfy (reflexivity, symmetry, transitivity)

II. If it is an equivalence relation,

  (i) how many equivalence classes it has
  (ii) a set containing one element from each equivalence class.

(6) The relation on $\mathbb{N}$ given by $a \sim b$ provided that $a \neq b$.

(7) The relation on $\mathbb{N}$ given by $a \sim b$ provided that a and b have no common divisors.

(8) The relation on $\mathbb{N}$ given by $a \sim b$ provided that $a - b = 2k$ for some integer $k \in \mathbb{Z}$.

(9) The relation on $\mathbb{Q}$ given by $a \sim b$ provided that a and b have the same denominator when written in least terms.

(10) The relation on $\mathbb{R}$ given by $a \sim b$ provided that $a < b$.

(11) The relation on $\mathbb{R}$ given by $a \sim b$ provided that $a \leq b$.

(12) The relation on $\mathbb{R}$ given by $a \sim b$ provided that $ab \geq 0$.

(13) The relation on $\mathbb{R}$ given by $a \sim b$ provided that $f(a) = f(b)$, when $f \colon \mathbb{R} \to \mathbb{R}$ is each of

   (a) $f(x) = x^2$
   (b) $f(x) = 1$
   (c) $f(x) = x$
   (d) $f(x) = \sin(x)$.

(14) The relation on $\mathbb{R}$ given by $a \sim b$ provided that $a - b \in \mathbb{Q}$.

(15) The relation on the set of triangles given by $T \sim S$ provided that T and S are *similar* (in the sense that you learned in high school geometry class).