Unofficial Practice Questions for Linear Algebra Exam 1

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May 24, 2018

1. For each of the following (italicized) terms,

   - Give **two** different, equivalent, precise mathematical characterizations for it; and
   - State which one of your characterizations is the definition.

   (a) The *dimension* of a linear subspace $V \subseteq \mathbb{R}^n$.

   (b) A system of equations is *inconsistent*.

   (c) The vectors $(\vec{v}_1, \ldots, \vec{v}_k)$ form a *basis* for $\mathbb{R}^n$.

   (d) The linear transformation $T : V \to W$ is *injective*.

2. State whether each statement is True or False, and provide either a short, one or two line proof, or else a counterexample.

   (a) If $(\vec{v}_1, \ldots, \vec{v}_k)$ are linearly *dependent* in $\mathbb{R}^n$, then for any linear transformation $T : \mathbb{R}^n \to W$, the vectors $(T\vec{v}_1, \ldots, T\vec{v}_k)$ are linearly dependent in W.

   (b) Let A be an $n \times m$ matrix and $\vec{b} \in \mathbb{R}^n$ such that the equation $A\vec{x} = \vec{b}$ has at least one solution. Then the set $\{\vec{x} \in \mathbb{R}^m : A\vec{x} = \vec{b}\}$ is a subspace of $\mathbb{R}^m$.

   (c) If A is an $n \times m$ matrix such that $\ker(A) = \{0\}$ and A is surjective, then $n = m$.

   (d) If $T : U \to V$ and $S : V \to W$ are linear transformations such that $S \circ T = 0$, then $\ker(S) \subseteq \operatorname{im}(T)$.

   (e) If A is a 2×2 matrix such that $\operatorname{rank}(A) \geq 1$, then for any $n \in \mathbb{N}$, $A^n \neq 0$.

   (f) *(Extra credit)* If $X \subseteq \mathbb{R}^n$ is a set such that:

   - $\vec{0} \in X$,
   - for all $\vec{x} \in X$ and $a \in \mathbb{R}$, $a\vec{x} \in X$,

   then for all $\vec{x}, \vec{y} \in X$, $\vec{x} + \vec{y} \in X$, and hence X is a subspace of $\mathbb{R}^n$.

3. (a) Let $A = \begin{bmatrix} 1 & 0 & 1 & 24 \\ 4 & 6 & 5 & 0 \end{bmatrix}$. Find a basis for $\ker(A)$.

   (b) Find a 4×2 matrix B such that $\text{im}(B) = \ker(A)$.

   (c) Which has larger rank, AB or BA?

4. Let $\mathcal{P}_2$ be the vector space consisting of all polynomials of degree at most 2 in the variable x.

   (a) Find a basis $\mathfrak{B}$ for $\mathcal{P}_2$ such that $[x^2]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $[x]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $[1]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

   (b) Consider the linear transformation

$$\begin{aligned} T : \mathcal{P}_2 &\to \mathcal{P}_2 \\ f(x) &\mapsto f'(x). \end{aligned}$$

   Find the matrix of T with respect to your basis $\mathfrak{B}$.

   (c) Find a basis for $\ker(T)$, and express it with respect to $\mathfrak{B}$.

   (d) Find a basis for $\text{im}(T)$, and express it with respect to $\mathfrak{B}$.

5. Consider the augmented matrix $C = \left[\begin{array}{ccc|c} 9 & -4 & -1 & j \\ 7 & 2 & 0 & k \\ -1 & 5 & 1 & \ell \end{array}\right]$ for the system of equations $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 9 & -4 & -1 \\ 7 & 2 & 0 \\ -1 & 5 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} j \\ k \\ \ell \end{bmatrix}.$$

   (a) Show that if $\vec{x}, \vec{y}$ are two solutions to this equation, then $\vec{x} - \vec{y} \in \ker(A)$.

   (b) For which values of j, k, ℓ is the set $\{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{b}\}$ a subspace of $\mathbb{R}^3$?

6. A matrix A is said to have a *square root* provided that there is some matrix B such that $B^2 = A$. We then write $B = \sqrt{A}$.

   (a) Show that if A is invertible and $A = \sqrt{A}$, then $A = I$.

   (b) Show that it is possible for a matrix to have more than one square root, by exhibiting two square roots for $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

   (c) Find a matrix which has no square roots.

   (d) Show that $\ker(\sqrt{A}) \subseteq \ker(A)$.