Linear Algebra Problems

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1. Let $k \neq 0$ be a scalar. Without using row reduction, compute the inverse of

$$D = \begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{bmatrix}.$$

2. Let $\vec{w} \neq \vec{0}$ be a vector in $\mathbb{R}^n$. Without using row reduction, compute the inverse of $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{x}) = 2\left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} - \vec{x}$.

3. Let $\theta \in \mathbb{R}$. Without using row reduction, compute the inverse of

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

4. Let k, ℓ be scalars.

   (a) Without using row reduction, compute the inverse of $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.

   (b) Without using row reduction, compute the inverse of $B = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}$.

   (c) Now compute the inverse of $C = \begin{bmatrix} 1 + k\ell & k \\ \ell & 1 \end{bmatrix}$.

5. For each of the following matrices, describe the image and kernel. If there is an inverse, compute it.

   (a) $E = \begin{bmatrix} 2 & -2 \end{bmatrix}$. (b) $F = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. (c) $H = \begin{bmatrix} 9 & -4 & -1 \\ 7 & 2 & 0 \\ -5 & 5 & 1 \end{bmatrix}$.

6. Billy's sheep live in a square pen. The pen is made of rope stretched between four posts, which are placed at the corners $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$. Every sheep needs a certain amount of area to graze, and in this pen, Billy is able to keep 18 sheep.

(a) In the night (when the sheep are sleeping in the barn) a windstorm pushes the posts to new positions, transforming them by the matrix

$$W = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}.$$

(Miraculously, the ropes do not snap.)

Since the pen now has a vastly different shape, Billy is worried that he may have to sell some of his sheep. How many sheep can Billy keep in the new pen?

(b) The next night, a tornado pushes the pen again, this time transforming the positions of the posts by

$$U = \begin{bmatrix} \cos(140°) & -\sin(140°) \\ \sin(140°) & \cos(140°) \end{bmatrix}.$$

How many sheep can Billy keep in the new pen?

(c) The next night, a snowstorm causes the ropes to cool and contract, transforming the positions of the posts by

$$T = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

How many sheep can Billy keep in the new pen?

(d) The next day, Billy tries to shear his sheep, but accidentally shears their pen instead, transforming it by

$$S = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

How many sheep can Billy keep in the new pen?

(e) The next night, one of the posts falls out. How many sheep can Billy keep in the new pen?

(f) The next night, a tidal wave pushes the pen, transforming it by

$$P = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

How many sheep can Billy keep in the new pen?

(g) The next night, a magnetic storm changes everything on the farm around in confusing ways. The posts of Billy's enclosure are transformed by

$$Q = \begin{bmatrix} 7 & 10 \\ 9 & 11 \end{bmatrix}.$$

How many sheep can Billy keep in the new pen?