Rank and Nullity Problems

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1. Let E be a square matrix. Prove that $\ker(E) \subseteq \ker(E^2)$.
2. Let C be a 5×2 matrix and D be a 2×5 matrix.
   Suppose $DC = I_2$.
   (a) Prove that $\dim(\ker(D)) \leq 3$.
   (b) Prove that $CD \neq I_5$
3. Let A be a 4×4 matrix such that $\operatorname{rank}(A^2) = 3$.
   (a) Prove that $\dim(\ker(A)) \leq 1$.
   (b) Prove that $\dim(\ker(A)) \geq 1$.
   (c) Prove that $\operatorname{rank}(A) = 3$.

Some facts to know about kernels and images

- **Images and kernels of products.**
  Let A and B be matrices such that the product BA makes sense. A couple facts to know:

  - $\ker(A) \subseteq \ker(BA)$.
    Proof. Fix $\vec{v} \in \ker A$. We know $A\vec{v} = \vec{0}$, so $BA\vec{v} = B\vec{0} = \vec{0}$. Therefore $\vec{c} \in \ker BA$.
  - $\operatorname{im}(BA) \subseteq \operatorname{im}(B)$.
    Proof. Fix $\vec{w} \in \operatorname{im}(BA)$. We know that there exists a vector $\vec{u}$ in the domain of BA such that $\vec{w} = (BA)\vec{u}$. Then consider the vector $A\vec{u}$. We see $\vec{w} = (BA)\vec{u} = B(A\vec{u}) \in \operatorname{im}(B)$.

- **Images and kernels and invertibility.**
  A square matrix is invertible if and only if its kernel is $\{0\}$ if and only if its image is the entire space. In other words, the rank is the maximum possible and the nullity is zero.
  (Any matrix with maximum possible rank is said to be *full-rank*.)

- **Images and kernels are of lesser or equal dimension to the spaces they are contained in.**
  This follows from the general fact about subspaces, but is often useful for solving complicated problems.

- **Rank-Nullity Theorem.**
  Suppose A is an $n \times m$ matrix (so the corresponding linear transformation is $\mathbb{R}^m \to \mathbb{R}^n$). Then
  $$\operatorname{rank}(A) + \operatorname{nullity}(A) = m, \text{ where}$$
  $$\operatorname{rank}(A) = \dim(\operatorname{im}(A)),$$
  $$\operatorname{nullity}(A) = \dim(\ker(A))$$

Remember: Invertible matrices have to be square. So if you have an $n \times m$ matrix A and an $m \times n$ matrix B such that $BA = I_m$, we don't say A is invertible.
Remember: Draw pictures to show which spaces are contained in which, and remember to justify every containment! Write next to your pictures the inequalities you get with dimensions of different spaces. I do the pictures in my head, but I had to do many problems with the pictures drawn out before I got good at that.
Remember: All the spaces in sight (images, kernels) are vector spaces, so when we say something is a *sub*space, we're really just making the claim that it's a subset, and you should try to prove containment in a set-wise mindset (Fix (arbitrary) $x \in X$, show $x \in Y$.)