I Think This is Related to JNF?

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We will answer the question "if A is an $n \times n$ matrix, what are the possibilities for the sequence $\{\text{rank}(A^k)\}_{k=0}^{\infty}$?" For the rest of the worksheet, let A be an $n \times n$ matrix.

1. (a) Prove that for all $k \geq 0$, $\ker(A^k) \subseteq \ker(A^{k+1})$.
   (b) Prove that for all $k \geq 0$, $\text{im}(A^k) \supseteq \text{im}(A^{k+1})$.
   (c) Conclude that $\{\text{rank}(A^k)\}$ is a nonincreasing sequence.

2. Suppose $\text{rank}(A^2) = n - 1$.
   (a) Prove that $\text{rank}(A) = n - 1$.
   (b) Prove that $\text{rank}(A^j) = n - 1$ for all $j \geq 1$.

3. Suppose $\text{rank}(A^k) = \text{rank}(A^{k+1})$. Prove that $\text{rank}(A^{k+j}) = \text{rank}(A^k)$ for all $j \geq 0$.

4. Consider the sequence $\{\text{rank}(A^k) - \text{rank}(A^{k+1})\}_{k=0}^{\infty}$.
   (a) Prove that $\text{rank}(A^k) - \text{rank}(A^{k+1}) = \dim(\ker(A) \cap \text{im}(A^k))$ for all $k \geq 0$.
   (b) Prove that $\{\text{rank}(A^k) - \text{rank}(A^{k+1})\}$ is a non-increasing sequence.
   (c) Conclude that the sequence $\{\text{rank}(A^k)\}$ is convex; i.e., for $k \geq 1$,
   $$\text{rank}(A^k) \leq \frac{\text{rank}(A^{k+1}) + \text{rank}(A^{k-1})}{2}.$$
   (d) Reprove problems (2) and (3) using the above fact.

5. Suppose $n = 10$. Construct an $n \times n$ matrix such that
   (a) $\text{rank}(A^k) = 10 - k$ for $0 \leq k \leq 10$.
   (b) $\text{rank}(A^0) = 10$, $\text{rank}(A^1) = 8$, and $\text{rank}(A^k) = 9 - k$ for $1 \leq k \leq 9$.
   (c) $\text{rank}(A^0) = 10$, $\text{rank}(A^1) = 6$, $\text{rank}(A^2) = 3$, $\text{rank}(A^3) = 1$, $\text{rank}(A^4) = 0$.

6. Show that for any nonincreasing convex sequence of integers $0 \leq r_k \leq n$, there exists an $n \times n$ matrix A such that $\text{rank}(A^k) = r_k$.