Endogenous Information Acquisition in Bayesian Games with Strategic Complementarities*

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Abstract

This paper studies covert (or hidden) information acquisition in common value Bayesian games of strategic complementarities. Using the supermodular stochastic order to arrange the structures of information increasingly in terms of preferences, we provide novel, easily interpretable conditions under which the value of information is globally convex, and study the implications in terms of the equilibrium configuration. Increasing marginal returns to information lead to extreme behavior in that agents opt either for the highest or the lowest quality signal. This result explains the complete information game as an endogenous outcome, and suggests the possibility of quite asymmetric information equilibria even in games that are themselves symmetric. Our analysis also enlightens the effect of information on players’ behavior.

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1 Introduction

The emergence of information economics as a major subfield of microeconomics, grounded on solid theoretical foundations, owes much to the theory of Bayesian games (Harsanyi, 1967-69 and Mertens and Zamir, 1985). The basic formulation of this class of games, adopted by most economic applications, posits an exogenously given structure of information. An unknown payoff-relevant parameter is part of the structure of the game and the players receive some exogenous partial information (or message) that guides their play in the game. This message is usually costless and its reliability or quality is fully out of the control of the players.

Yet, in many of the natural economic settings that invoke this class of games, it would be more appropriate to postulate that the players have the option of acquiring further information on the unknown parameter, beyond what is readily available, provided they pay the corresponding cost. Consider for instance price-setting firms that compete in an industry where demand is known up to a parameter whose prior distribution is common knowledge. To deal with this uncertainty, some of the firms might consult a market research firm, while others might search for information on the Internet. The first group is likely to get better—and thus expensive—advice. The second group can invest more or less time on the Internet—at varying opportunity cost. Regardless of the method used to collect information, the acquisition process is costly and the firms will choose to acquire more information if the benefit in terms of better pricing decisions exceeds its cost. However, an important complication for these firms is that, in a strategic setting, better information need not translate into higher profits at the ensuing Bayesian equilibrium.\footnote{In oligopoly models, endogenous information acquisition is studied by Hwang (1993), Hauk and Hurkens (2001), Jansen (2008), Lee (1992), Li, McKelvey and Page (1987), Vives (1988), and Dimitrova and Schlee (2003), among others. These studies assume specific functional forms for the payoffs and, in most cases, Gaussian distributions for the signals. In mechanism design this topic is addressed by Bergemann and Välimäki (2002) and Persico (2000). Though closer in spirit to the present paper, their settings are quite different from ours.}

In an attempt to provide a general analysis of such situations, this paper endogenizes information acquisition in a particular class of games with incomplete information. We focus on common value Bayesian supermodular games in which (i) actions are strategic complements; (ii) for each player, own action and the unknown are complements; and (iii) interim beliefs increase in messages. Without endogenous information, the framework of the present paper would be a special case of that of Van Zandt and Vives (2007). For a broad class of Bayesian games with strategic complementarities, they prove existence of a greatest and a least Bayesian Nash equilibrium, both of which in
strategies that are monotone increasing in type (see also Van Zandt, 2010). To this basic building block, we add an initial period where players decide on the quality of their own signal in light of a given cost function for signal quality. This presumes tacitly that there is an information sector characterized by a production function for information as a good.

The formal set-up is as follows. Consider \( n \) agents who face two sequential decisions: First, with the cost of information being common knowledge, each player chooses how much information to buy independently of, and simultaneously with, his opponents. Then, after observing the realization of his purchased signal (message), but neither the messages received nor the quality of information acquired by the other agents, each player selects an action in the Bayesian game that follows, the latter being of the sort studied by Van Zandt and Vives (2007). To solve for the Bayesian information Nash equilibrium of the whole game we follow an approach used by Hauk and Hurkens (2001) to study covert information acquisition in a Cournot game. We start the analysis with an artificial game that models the second decisions of the players, for which we adopt the findings of Van Zandt and Vives (2007). Then we move back to integrate the information acquisition phase and study the equilibrium of the entire game. While this may be viewed as a two-stage game, there is no observability of "first-stage" or information decisions before entering the "second stage". In other words, when selecting an action for the "second stage", the only signal quality a player knows is his own, whence the designation as "covert information acquisition".

To ensure that more information is always valuable (when acquisition costs are not taken into account), we use the supermodular stochastic order to rank information structures in terms of players' preferences, so that "informativeness" is related to a signal being "more correlated" with the state. Athey and Levin (2001) use the same order to compare preferences for information in monotone decision problems, where the posterior beliefs induced by the signal can be ordered so that higher actions are chosen in response to higher signal realizations. The present analysis extends their finding to unilateral deviations in information acquisition in models with strategic interactions.\(^2\) We also investigate how information affects agents' behavior regarding strategies in the second (Bayesian) phase of the game. Our results suggest that "more informative" signals lead agents to take more extreme actions in the second stage. The intuition is easy to grasp: When the quality of the signal is higher, agents have more faith in the messages they receive and this encourages them to make more extreme decisions later on.

The central part of the present paper is concerned with the second order properties of the value of (covert) information as a way to characterize a class of models that possess extreme pure strategy Bayesian information equilibria. This class of models is characterized by a novel assumption of convexity (resp., concavity) of the information structure in the supermodular order. The interpretation of this notion of convexity (resp., concavity) is quite natural for the setting at hand: It says that a higher signal quality raises informativeness with increasing (resp., decreasing) returns. Under this notion of convexity, in this class of information acquisition games, each player will always choose to acquire either the minimal or the maximal quality signal, thus reducing the otherwise complex equilibrium existence issue to one in a two-action (matrix) game.\(^3\)

In order to broaden readership, we also provide a basic illustration of the underlying mechanism using a standard beauty contest game, different versions of which have recently been analyzed in different settings (e.g., Hellwig and Veldkamp, 2009 and Myatt and Wallace, 2012). This particular formulation relies on quadratic payoffs and Gaussian information, and allows us to motivate the role of the stochastic supermodular order in our analysis. In addition, this example serves as a natural setting in which to conveniently bring out the role of the scale used to measure information on the resulting second order effects of information acquisition.

The second order implications of information acquisition have been extensively investigated in the (single-player) case or decision theory. Several studies show that the value of information often exhibits increasing marginal returns over some range, specifically near zero. In particular, Radner and Stiglitz (1984) and Chade and Schlee (2002) offer a detailed investigation of this convexity of the value function in a broad class of information acquisition problems.\(^4\) Our analysis might be seen as an extension of their results to a multi-agent framework, using the supermodular order to derive global conditions for convexity. Confirming their finding, we also elaborate on the difficulties involved in getting the value of information to be globally concave.

One of the broad-based possible uses of the present setting is as a framework for costly rationality in strategic settings. Proponents of such a framework argue that some of the well-known violations of the hyper-rationality paradigm, such as those commonly observed in some laboratory experiments, might be better explained by a recognition that, in many environments, good decisions would require the acquisition of costly information, rather than the more common postulate that economic agents

\(^3\)Motivated by different issues in economic dynamics, related notions of stochastic convexity of integrals with respect to parameters of transition probabilities appear in Amir (1996) for first order stochastic dominance and Amir (1997) for second order stochastic dominance.

\(^4\)Among other economic applications where this convexity result emerges, Dimitrova and Schlee (2003) study information acquisition and entry threat in a monopoly market. As we do here, their paper highlights the natural drawbacks of the non-concavity result for modelling strategic interactions. See also Bagli (2014).
are inherently irrational or boundedly rational. Radner (2004) discusses this premise in some detail from a decision and game-theoretic perspective. Similar ideas have also quite naturally emerged and become influential in macroeconomic theory under the heading of "rational inattention" (Sims, 2003; 2010). The setting of a Bayesian framework with endogenous costly signals is clearly a natural candidate in economic settings characterized by the juxtaposition of strategic behavior on the one hand and informational scarcity or processing limitations on the other hand.\(^5\)

The increasing returns to scale in information acquisition is a central feature to our approach to the existence of equilibria for the overall game. From the perspective of costly rationality in game theory, the extreme behavior of the players might be seen as justifying the familiar, complete information games as an endogenous outcome in cases where maximal signal quality corresponds to complete information and information costs are not prohibitive. Another notable outcome is the possibility of maximally asymmetric equilibria even in games with ex ante identical players, where one player ends up with maximal, and the other player with minimal, signal quality. We illustrate such a possibility via an example, which neatly underscores the differences of our setting relative to a non-strategic setting (where identical agents would always select the same optimal solution). Furthermore, this example provides a strategic setting in which players with identical priors might "agree to disagree" as the outcome of a Bayesian game with endogenous information acquisition. While this does not contradict the well known results on the impossibility of "agreeing to disagree" (Aumann, 1976), it is a worthwhile feature to point out.

The rest of the paper is organized as follows. The next section describes the model and the notion of equilibrium. Section 3 offers an analysis of a common beauty contest as motivating example; its purpose is to provide some preview of, and intuition for, the approach and the results. Section 4 analyses the general game; it elaborates on the comparison between information structures and on the second order effects on expected payoffs, and ends by showing existence of (extreme) pure-strategy equilibria; it also enlightens the effect of information on players’ behavior. Section 5 concludes, and the last section includes all the proofs.

2 The analytical framework

In this section we set-up a model of hidden (or covert) information acquisition for a common value Bayesian game of strategic complementarities. This class of games includes various models of

\(^5\)Neyman (1991) argues that the framework of Bayesian games is appropriate to contrast a player’s welfare as his information changes in an unilateral way. He also establishes that increasing information in such a setting can never hurt the said player. In addition, Bassan, Gossner, Scarsini and Zamir (2003) describe a class of interactive decision models where all the agents are better off having more information.
oligopolistic competition, arms races, bank runs, R&D, among others.

To clarify notation, $\geq$ indicates partial orders in general, including the natural order on the reals. All Cartesian products (including uncountable ones) are tacitly ordered by the product order. When we say "greater than" and "increasing" it is in a weak sense. Measurability of all functions is always (tacitly) with respect to the relevant Borel sigma-algebra. We distinguish random variables from their realizations by using wiggles, e.g., $\omega$ denotes a realization of the random variable $\tilde{\omega}$.

For further reference, we assume the conditions in the next two subsections (Assumptions 1, 2 and 3) hold throughout the paper.  

2.1 Payoffs

The set of players is $N = \{1, ..., n\}$, with a typical element indexed by $i$. Each agent $i$ chooses an action $a_i$ from a compact metric lattice $A_i$.  

His payoff is given by

$$u_i(a_i, a_{-i}, \omega): A_i \times A_{-i} \times \Omega \to \mathbb{R}$$

where $a_{-i} \in A_{-i} \equiv \prod_{j \neq i, j \in N} A_j$ denotes the vector of other agents’ actions, and $\omega \in \Omega \subseteq \mathbb{R}$ is the realization of an exogenous payoff relevant random variable. We will refer to $\tilde{\omega}$ as the state of the world or the fundamental. Agents choose their actions before the state of the world is realized. The prior distribution of $\tilde{\omega}$, $H(\cdot)$, is common knowledge among them. The first assumption, that payoffs are Caratheodory functions, is standard.

**Assumption 1.** For all agent $i$ in $N$, we assume (i) $u_i(a, \cdot): \Omega \to \mathbb{R}$ is measurable $\forall a \in A \equiv \prod_{i \in N} A_i$; (ii) $u_i(\cdot, \omega): A \to \mathbb{R}$ is jointly continuous $\forall \omega \in \Omega$; and (iii) $u_i$ is bounded.

Complementarities arise in our economy through two different channels. First, the incremental returns of agent $i$ with respect to own action increase in the action profile of the others. Second, the marginal profitability of an increase in a player’s action increases in the fundamental. To fix ideas, the reader might keep in mind a Bertrand oligopoly with differentiated substitute products and demand uncertainty (further studied in Section 3). Assumption 2 formalizes these conditions.

**Assumption 2.** For all agent $i$ in $N$, we assume (i) $u_i$ is supermodular in $a_i$; and (ii) $u_i$ has strictly increasing differences in $(a_i; a_{-i}, \omega)$.

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6The assumptions we enumerate in Sections 2.1 and 2.2 are stronger than the requirements in Van Zandt and Vives (2007). This allows us to use a somewhat stronger version of their main results in Section 4.1, the added strength being quite useful in our setting.

7Although each player’s actions may be multi-dimensional, we do not use bold letters to write $a_i \in A_i$. We reserve bold letters to denote a profile of actions of all players, $a \in A$, and the profile of actions of all players other than $i$, $a_{-i} \in A_{-i}$. 

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Increasing differences are often interpreted as formalizing the notion of (Edgeworth) complementarity. Although none of our results rely on differentiability, if \( A_i \) is an interval on \( \mathbb{R}^k \) and \( u_i \) is twice continuously differentiable, these conditions adopt the following well-known simple form.

Assumption 2’. For all agent \( i \) in \( N \), we assume (i’) \( \frac{\partial^2 u_i}{\partial a_{im}\partial a_{il}} \geq 0 \) \( \forall 1 \leq m < l \leq k_i \); and (ii’) \( \frac{\partial^2 u_i}{\partial a_{im}\partial a_{jl}} \geq 0 \) \( \forall i \neq j \), \( \forall 1 \leq m \leq k_i \), \( \forall 1 \leq l \leq k_j \), and \( \frac{\partial^2 u_i}{\partial a_{im}\partial \omega} \geq 0 \) \( \forall 1 \leq m \leq k_i \).

We next describe the information structure of the game.

2.2 Signals and posteriors

Player \( i \) can acquire additional information about the state of the world by purchasing a noisy signal. Let \( S_i \subseteq \mathbb{R} \), with typical element \( s_i \), denote the set of all possible signal realizations that player \( i \) might observe. \( S_i \) is assumed to be a compact interval or a finite subset of the reals. We assume that all messages in \( S_i \) are non-trivial. Following the related literature, player \( i \) gathers information by choosing from a family of joint distributions \( \{ F (s_i, \omega; \alpha_i) \} \) indexed by \( \alpha_i \in [\underline{\alpha}, \overline{\alpha}] \subseteq \mathbb{R} \). Each \( \alpha_i \) is related to a statistical experiment, and increasing \( \alpha_i \) raises informativeness in a way that we shall formalize later. For the purpose of our analysis, we can think of \( \alpha_i \) as the quality of player \( i \)'s signal. Conditional on \( \omega \), players' signals are assumed independent.

In our setting, \( C_i : [\underline{\alpha}, \overline{\alpha}] \rightarrow \mathbb{R}_+ \) denotes the cost of information acquisition for player \( i \). We assume \( C_i (\cdot) \) is increasing and \( C_i (\underline{\alpha}) = 0 \), i.e. \( \underline{\alpha} \) is the quality of the signal that the players can get for free. Without loss of generality, we normalize the cost of information so that\(^9\)

\[
C_i (\alpha_i) = \alpha_i - \underline{\alpha}, \text{ for all } i = 1, 2, \ldots, n. \tag{2}
\]

The reason this linear cost function is without loss of generality is that, starting with any strictly increasing and continuous cost function \( \tilde{C}_i (\tilde{\alpha}_i) \) for quality of information \( \tilde{\alpha}_i \), we can consider instead a change of variable \( \alpha_i = \tilde{C}_i^{-1} (\tilde{\alpha}_i) \). This leads to the linear cost function (2) but also changes the structures of information accordingly. We are tacitly assuming that the given family \( \{ F (s_i, \omega; \alpha_i) \} \) already reflects such a change of variable. Thus, assuming a linear cost function may be seen as imposing a particular way of measuring information quality. This is a convenient measure for reasons that will become clear later.

We assume that signals are independent conditional on \( \omega \). Let \( s \in S \equiv \Pi_{i \in N} S_i \) and \( \alpha \in [\underline{\alpha}, \overline{\alpha}]^n \) denote the realization of a vector of signals and a quality profile, respectively. Each \( n \)-tuple \( \alpha \in \mathbb{R}^n \).

\(^8\)Consistency requires \( E[F (s_i, \omega; \alpha_i) | \omega; \alpha_i] = H (\omega) \) for all \( \alpha_i \in [\underline{\alpha}, \overline{\alpha}] \).

\(^9\)A similar approach is used by Ganuza and Penalva (2006).
\([\alpha, \overline{\alpha}]\) induces a joint cumulative distribution function (cdf) \(F(s, \omega; \alpha) : S \times \Omega \to [0, 1]\).\(^{10}\) Given \(\alpha\), we let \(F(s | \omega; \alpha)\) denote the cdf of \(\overline{S}\) conditional on \(\omega\), and let \(F(s_i, \omega | s_i; \alpha)\) represent agent \(i\)'s interim beliefs when he receives a message \(s_i\).\(^{11}\)

We assume the posterior distribution \(F(s | \omega; \alpha)\) and the interim beliefs \(F(s_i, \omega | s_i; \alpha)\) satisfy the following standard monotonicity condition.

**Assumption 3.** For every agent \(i\) in \(N\), we assume (i) \(F(s | \omega; \alpha)\) is first order stochastically increasing in \(\omega\); and (ii) \(F(s_i, \omega | s_i; \alpha)\) is first order stochastically increasing in \(s_i\).

By definition, Condition (i) is equivalent to say that, for all increasing sets \(E\) in \(S\),

\[
\int_S 1_E \, dF(., \omega; \alpha) \leq \int_S 1_E \, dF(., \omega'; \alpha) \quad \text{for all } \omega > \omega'
\]

where \(1_E\) denotes the characteristic function of the set \(E\).\(^{12}\) A similar statement holds for Condition (ii). Thinking of Nature as an additional player of type \(\omega \in \Omega\), these conditions mean that players’ interim beliefs increase in messages in the sense of first order stochastic dominance.

### 2.3 The game and the equilibrium concept

The timing of the game is as follows. During stage I, each agent \(i\) chooses \(\alpha_i \in [\alpha, \overline{\alpha}]\) independently and concurrently with other players; and at stage II, he observes the realization of his signal, \(s_i\), and chooses an action \(a_i \in A_i\)—he does not observe the signals of the others. We denote this game by \(\Gamma\). Since the only information revealed to the players at stage II are their own messages, we can think of \(\Gamma\) as a one shot game. We distinguish between stages I and II because it clarifies our analytical approach.\(^{13}\)

A pure strategy for agent \(i\) in \(\Gamma\) consists of a pair \((\alpha_i, \sigma_i)\), where \(\alpha_i \in [\alpha, \overline{\alpha}]\) is the quality acquired at stage I and \(\sigma_i : S_i \to A_i\) is a Borel measurable function that maps messages into actions. We let \(\Sigma_i\) denote the set of all strategies \(\sigma_i\) for player \(i\).

\(^{10}\)We shall write integrals in the style of Lebesgue-Stieltjes integration throughout the paper, as our assumptions on the underlying probability measures are more conveniently expressed on the associated cdf's. Given the need to use various marginal and conditional cdf's derived from \(F(s, \omega; \alpha)\), we adopt a common abuse of notation in the literature in denoting all of these conditional cdf's by the same function \(F\), the distinction between them being reflected only in that integration variables are always to the left, and conditioning arguments to the right, of the vertical bar.

\(^{11}\)Since \(S\) and \(\Omega\) are both subsets of the reals, the various conditional and marginal cdf's of \(F(s, \omega; \alpha)\) all clearly exist, and (tacitly) satisfy the standard assumptions in probability theory throughout: Monotonicity and right continuity with respect to integration variables and Borel measurability with respect to conditioning variables.

\(^{12}\)A set \(E\) in \(R^n\) is increasing if \(x' \in E\) whenever \(x \in E\) and \(x' \geq x\).

\(^{13}\)Persico (2000) uses a similar setting to compare the incentives to acquire information in first and second price auctions; Hauk and Hurkens (2001) and Vives (1988) to study information acquisition in Cournot games.
Given a strategy profile \((\alpha, \sigma)\), with \(\alpha \in [\underline{\alpha}, \overline{\alpha}]^n\) and \(\sigma \in \Sigma \equiv \prod_{i \in N} \Sigma_i\), let \((\alpha_{-i}, \sigma_{-i})\) denote the profile where players other than \(i\) follow their corresponding strategies at \((\alpha, \sigma)\). We next introduce the equilibrium notion for the game \(\Gamma\) used in this paper.

**Definition 1** A pure strategy profile \((\alpha^*, \sigma^*) \in [\underline{\alpha}, \overline{\alpha}]^n \times \Sigma\) is a Bayesian Nash equilibrium with endogenous information of \(\Gamma\) if

\[
(\alpha^*_i, \sigma^*_i) \in \arg \max_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}], \sigma_i \in \Sigma_i} \left\{ \int_{\mathbb{S}, \Omega} u_i(\sigma_i(s_i), \sigma^*_{-i}(s_{-i}), \omega) \, dF(s, \omega; \alpha_i, \alpha^*_{-i}) - C_i(\alpha_i) \right\},
\]

for each player \(i \in N\).

Although this paper deals solely with the game \(\Gamma\), the method of analysis used invokes the related two-stage game \(\Gamma_{II}\), obtained from \(\Gamma\) via the one modification that each player \(i\) observes the entire choice vector \(\alpha\) before choosing a second-stage strategy \(\sigma_i, i = 1, \ldots, n\). Let \(\Gamma_{II}(\alpha)\) denote the subgame of \(\Gamma_{II}\) starting at stage 2, given the observed first stage choices \(\alpha\). Thus \(\Gamma_{II}(\alpha)\) is the exogenous information counterpart of \(\Gamma\), wherein the profile of information \(\alpha\) is common knowledge. In other words, for each given \(\alpha\), \(\Gamma_{II}(\alpha)\) is a Bayesian supermodular game, exactly as defined by Van Zandt and Vives (2007). The game \(\Gamma_{II}(\alpha)\) should not be viewed as the second stage of the game \(\Gamma\), due to \(\alpha\) being common knowledge in the former but never in the latter. Nevertheless, following Hauk and Hurkens (2001), the Bayesian Nash equilibria of \(\Gamma_{II}(\alpha)\) for a given \(\alpha\) will turn out to form a key step in the existence and characterization of the equilibria of the game \(\Gamma\). The precise connection is that if \((\alpha^*, \sigma^*)\) is an equilibrium of \(\Gamma\) according to Definition 1, then \(\sigma^*\) is a Bayesian Nash equilibrium of \(\Gamma_{II}(\alpha^*)\).\(^{14}\)

While the game \(\Gamma\) can be thought of and treated as a one shot game, the information acquisition phase and the action-taking phase may actually be quite spaced apart in time, the crucial feature being rather the lack of observability of rivals’ information decisions and signals by any one agent. This lack of observability may be a natural consequence of a number of different factors depending on the setting at hand. Some of the more obvious factors include secretive behavior on the part of players in general, and natural segregation and lack of communication (as mandated by antitrust regulation) amongst competing firms in market settings.\(^{15}\) As noted by Hauk and Hurkens (2001),

\(^{14}\)By contrast, a perfect Bayesian equilibrium of the game \(\Gamma_{II}\) is a pair \((\bar{\alpha}, \bar{\sigma}(\cdot))\) such that \(\bar{\sigma}(\alpha)\) is a Bayesian Nash equilibrium of the game \(\Gamma_{II}(\alpha)\) for every possible choice \(\alpha\). In the literature on dynamic games, the latter are usually termed closed-loop equilibria, while the equilibria of \(\Gamma\) are called open-loop equilibria.

\(^{15}\)In reality, there are a many natural channels through which presumably secret information partially leaks to competitors, in what are usually referred to as spillovers, say in the literature on innovation. Informational spillovers would be rather difficult to model in the present setting, due to their ambiguities in terms of common knowledge.
although the game \( \Gamma_{II} \) can also be appropriate in modeling strategic information acquisition in some settings, it is somewhat difficult to envision how the vector \( \alpha \) could realistically be common knowledge. Indeed, public announcements by firms are often designed to manipulate rivals’ beliefs, and are thus unreliable to rivals. While industrial espionage can succeed in providing information about a competitor’s state of know-how, it typically would not do so in a manner consistent with common knowledge.

To facilitate the understanding of our abstract framework, we begin with a simple concrete example, which illustrates our general results in a commonly used beauty-contest game.

### 3 A motivating example

To motivate the rest of the paper, we present a simple version of the beauty-contest in the spirit of our approach. Versions of this model were previously used by Hellwig and Veldkamp (2009) and Myatt and Wallace (2012). The setting is also similar to the oligopoly models of Hauk and Hurkens (2001) and Vives (1988). Although the analysis can be easily extended to an arbitrary number of players, we restrict attention to only two, for expositional ease.

A player’s payoff depends on the proximity of his action, \( a_i \in \mathbb{R} \), to both the underlying state variable, \( \omega \), and the average group action \( \bar{a} = (a_1 + a_2)/2 \),

\[
u_i = \bar{a} - (1 - \gamma) (a_i - \omega)^2 - \gamma (a_i - \bar{a})^2, \quad i = 1, 2 \tag{3}\]

where \( \gamma \in (0, 1) \), and \( \omega \) is the realization of a random parameter drawn from a normal prior with mean \( \mu \) and variance \( \sigma^2 \). The important feature of this model is that the marginal returns of a player’s own action, increase with the actions of the other players and with the state \( \omega \).

Player \( i \) acquires information by choosing from a family of cdf’s \( \{F(s_i; \omega; \alpha_i)\} \) indexed by \( \alpha_i \in [0, 1] \). Given some \( \alpha_i \in [0, 1] \), we assume the joint distribution of \( (\tilde{s}_i, \tilde{\omega}) \) is the bivariate normal

\[
\mathcal{N}[A, B(\alpha_i)], \quad \text{with } A = (\mu, \mu) \text{ and } B(\alpha_i) = \begin{pmatrix} 1 & \alpha_i \sigma \\
\alpha_i \sigma & \sigma^2 \end{pmatrix}.
\tag{4}\]

The variance of \( \tilde{s}_i \) is normalized to 1, so as to highlight the fact that, in our setting, what really matters to player \( i \) is the correlation between own signal and the state, captured by \( \alpha_i \).\(^{16}\)

Player \( i \) decides how much information to acquire by choosing a correlation level \( \alpha_i \), at a corresponding cost \( C_i(\alpha_i) = \alpha_i \). We assume signals are independent conditional on \( \omega \). Given a profile of qualities \( \alpha \), our conditions imply that \( \mathbb{E}(\omega | s_i; \alpha_i) = \mu + \alpha_i \omega (s_i - \mu) \) and \( \mathbb{E}(s_i | s_i; \alpha) = \)

\(^{16}\)For this example, since players’ payoffs are both quadratic and supermodular, information could be modeled with a mean-variance order. We opted for correlation to better connect the example to our general framework.
The bivariate normal distribution has an additional nice feature: if \( \alpha_i = 0 \) then \( \bar{s}_i \) is independent of \( \bar{\omega} \) and from player \( i \)'s perspective the message is pure-noise; and if \( \alpha_i = 1 \) then the conditional variance of \( \bar{\omega} \) given \( s_i \) vanishes and the signal reveals the value of the fundamental with certainty. Moreover, as will be seen later, the correlation \( \alpha_i \) orders the family of signals according to the supermodular stochastic order, a key feature in our general analysis.

The timing of the game is as described before. In the first stage, participants select the quality of information. After observing the realization of their own signals, but neither the experiment selected nor the message received by the other player, each of them chooses a price. As observed by Hauk and Hurkens (2001), in this simple setting, it is quite easy to solve for the Bayesian information Nash equilibrium. First, we solve for the (unique) Bayesian Nash equilibrium in the second stage. Then, substituting players’ actions by the corresponding strategies in the utility functions, we consider the game where only quality levels need to be chosen.

Assume for the moment that participants acquired a profile of information \( \alpha = (\alpha_1, \alpha_2) \) at stage I, and focus on the game that follows. For each \( \alpha \in [0, 1]^2 \), it is well known that (for the present formulation) the equilibrium strategies in the second stage are affine with respect to messages. To get them, we will assume player \(-i\) follows a strategy \( \sigma_{-i} (s_{-i}) = a + b_{-i} (s_{-i} - \mu) \). Inserting this expression in (3) we get

\[
 u_i = \bar{\pi} - (1 - \gamma) (a_i - \omega)^2 - \gamma (a_i/2 - (a + b_{-i} (s_{-i} - \mu))/2)^2.
\]

If player \( i \) receives a message \( s_i \), its interim payoff is

\[
 E (u_i | s_i; \alpha) = M_i - (1 - (3/4) \gamma) a_i^2/2 + (1 - \gamma) (a_i \mu + \alpha_i v_\omega (s_i - \mu)) + \gamma a_i (a + b_{-i} (s_{-i} - \mu))/2.
\]

where the conditional expectation is calculated with respect to the Bayesian updated beliefs. In (5), \( M_i = E \left( \bar{\pi} - (1 - \gamma) \omega^2 - \gamma (a + b_{-i} (s_{-i} - \mu))^2 /4 | s_i; \alpha \right) \). We avoid the latter computation as \( M_i \) does not play any fundamental role in the analysis that follows. Computing the first order condition, its best-response is given by

\[
 \alpha_i = \frac{1}{2(1 - \bar{\gamma})} \left\{ 2 (1 - \gamma) \mu + \gamma a_i/2 + [2 (1 - \gamma) \alpha_i v_\omega + \gamma b_{-i} \alpha_{-i}/2] (s_i - \mu) \right\}.
\]

Substituting \( i \) by \(-i\), we can get the corresponding expression for the other player. Combining results, the equilibrium strategies (as a function of \( \alpha \)) are

\[
 \sigma_i (s_i) = \mu + v_\omega (1 - \gamma) \frac{2 (1 - (3/4) \gamma) + \gamma \alpha_i^2/2}{4 (1 - (3/4) \gamma)^2 - (\gamma \alpha_i \alpha_{-i})^2/4} \alpha_i (s_i - \mu), \quad i = 1, 2.
\]

Hence the sensitivity of player \( i \)'s strategy with respect to unexpected shocks \((s_i - \mu)\) increases in both \( \alpha_i \) and \( \alpha_{-i} \).
To attain the profile of information at equilibrium, we need to go one step back and find the conditions under which no player \(i\) has an incentive to deviate from \(\alpha_i\). Two issues deserve attention. First, since we model hidden information acquisition, if player \(i\) deviates at stage I from \(\alpha_i\) to \(\alpha'_i\), there is no strategic effect on the other player. Second, if player \(i\) selects \(\alpha'_i\) instead of \(\alpha_i\), it will use this information to update its interim pricing strategy in the second stage as follows

\[
\varphi_i(s_i; \alpha'_i) = \mu + v_\omega (1 - \gamma) \frac{2 (1 - (3/4) \gamma) + \gamma \alpha_{-i}^2/2}{4 (1 - (3/4) \gamma)^2 - (\gamma \alpha_i \alpha_{-i})^2/4} \alpha'_i(s_i - \mu) \tag{8}
\]

where (8) is obtained by substituting \(\sigma_{-i}(s_{-i})\) as defined in (7), in (6). Notice (8) depends on both \(\alpha'_i\) and \(\alpha_i\), the reason being that player \(i\) can improve his strategy after deviating but cannot affect the other players’ beliefs and these beliefs do affect \(i\)’s optimal behavior.

Let’s assume player \(-i\) follows the equilibrium strategy at stage II corresponding to the profile \(\alpha\). We let \(U_i(\alpha'_i; \alpha)\) denote player \(i\)’s highest expected profits if it deviates from \(\alpha_i\) to \(\alpha'_i\). Taking the unconditional expectation of \(\pi_i\) after substituting \(p_i\) by \(\varphi_i(s_i; \alpha'_i)\) and \(\sigma_{-i}(s_{-i})\) we get

\[
U_i(\alpha'_i; \alpha) = E(M_i | \alpha) + \mu^2 + \left(v_\omega (1 - \gamma) \frac{2 (1 - (3/4) \gamma) + \gamma \alpha_{-i}^2/2}{4 (1 - (3/4) \gamma)^2 - (\gamma \alpha_i \alpha_{-i})^2/4}\right)^2 \alpha_i^2.
\]

Here, player \(i\)’s highest expected payoff has two properties of interest: It is increasing and convex in \(\alpha'_i\). In other words, the marginal returns to information are positive and increasing.

Taking into account the cost of information acquisition, player \(i\) will not have any incentives to deviate from \(\alpha_i\) if \(\alpha_i \in \arg \max_{\alpha'_i \in [0,1]} \{U_i(\alpha'_i; \alpha) - \alpha'_i\}\). We conclude that a profile of information \(\alpha = (\alpha_1, \alpha_2)\) constitutes an equilibrium if it satisfies, simultaneously, the system

\[
\begin{align*}
\alpha_1 \in \arg \max_{\alpha'_i \in [0,1]} \left\{ E(M_1 | \alpha) + \mu^2 + \left(v_\omega (1 - \gamma) \frac{2 (1 - (3/4) \gamma) + \gamma \alpha_{-i}^2/2}{4 (1 - (3/4) \gamma)^2 - (\gamma \alpha_i \alpha_{-i})^2/4}\right)^2 \alpha_i^2 - \alpha'_i \right\} \\
\alpha_2 \in \arg \max_{\alpha'_i \in [0,1]} \left\{ E(M_2 | \alpha) + \mu^2 + \left(v_\omega (1 - \gamma) \frac{2 (1 - (3/4) \gamma) + \gamma \alpha_{-i}^2/2}{4 (1 - (3/4) \gamma)^2 - (\gamma \alpha_i \alpha_{-i})^2/4}\right)^2 \alpha_i^2 - \alpha'_i \right\}.
\end{align*}
\]

Since \(U_i(\alpha'_i; \alpha) - \alpha'_i\) is strictly convex in \(\alpha'_i\) and the constraint set is \([0,1]\), its argmax is either 0, 1, or both, for \(i = 1, 2\). Then any equilibrium candidate \(\alpha\) reflects extreme behavior regarding information acquisition: Players acquire the full information signal or no information at all. Here a maximal and a minimal information equilibrium always exist because the induced game at stage I is supermodular in the \(\alpha\)’s.

In this setting \(\alpha = (1, 1)\) is an information equilibrium if

\[
\frac{2 (1 - (3/4) \gamma) + \gamma \alpha_{-i}^2/2}{4 (1 - (3/4) \gamma)^2 - (\gamma \alpha_i \alpha_{-i})^2/4} \geq 1
\]
and $\alpha = (0, 0)$ constitutes an equilibrium if

$$v_\omega (1 - \gamma) \frac{2 (1 - (3/4) \gamma)}{4 (1 - (3/4) \gamma)^2} \leq 1.$$ 

Thus the complete information game emerges endogenously if either the prior is very uninformative ($v_\omega^2$ large) or there are strong complementarities in the second stage ($\gamma$ small). The opposite is true for the common uncertainty, no private information, game. In addition, if

$$v_\omega (1 - \gamma) \frac{2 (1 - (3/4) \gamma) + \gamma \alpha_i^2/2}{4 (1 - (3/4) \gamma)^2 - (\gamma \alpha_i \alpha_{-i})^2/4} \geq 1 \geq v_\omega (1 - \gamma) \frac{2 (1 - (3/4) \gamma)}{4 (1 - (3/4) \gamma)^2},$$

then the two extreme information profiles are both equilibria.

Several insights can be gleaned from this example. First, expected payoffs from deviating increase in the correlation between the signal and the fundamental. Thus players care about information because it allows a better match between pricing decisions and the level of demand. Given the complementarities in the model, this also allows a better match between the players’ strategies. Although the value of information is always positive, it is also costly and players sometime decide not to get information at all.

Second, the quality of the signal decreases player $i$’s optimal strategy when it receives low messages, and increases it for high messages. Formally,

$$\frac{\partial \varphi_i (s_i; \alpha_i')}{\partial \alpha_i'} = v_\omega (1 - \gamma) \frac{2 (1 - (3/4) \gamma) + \gamma \alpha_i^2/2}{4 (1 - (3/4) \gamma)^2 - (\gamma \alpha_i \alpha_{-i})^2/4} (s_i - \mu),$$

which is positive if $s_i \geq \mu$ and negative otherwise. The intuition behind this behavior is simple: When player $i$ acquires a more precise signal, he puts more trust in the information received, and this encourages him to make more extreme decisions at stage II.

Third, the convexity of the payoffs induces players to behave in an extreme fashion with respect to information acquisition: Either pick the full information signal or remain fully uninformed. In addition, when both $(1, 1)$ and $(0, 0)$ are equilibria, the former is Pareto-preferred to the latter.

An important point, to which we shall return later on, is that the curvature of the value of information crucially depends on the way in which one actually measures information. In particular, if we let the agents in this model select information structures via the coefficient of determination, $\alpha_i^2$, instead of the correlation, $\alpha_i$, then $U_i$ becomes linear in own information $\alpha_i$. In other words, as this example illustrates, the curvature of $U_i$ is quite sensitive to the way in which we index information structures. However, in line with some previous work, our results below show that the general conditions that lead to convex information value are much weaker than the counterparts needed to guarantee concavity.

In short, the aim of this paper is to explore the extent to which the main features of this example may be construed as robust properties of general Bayesian games with endogenous information.
4 The information acquisition game

This section has four (related) goals. First, we investigate the exogenous information counterpart of $\Gamma$, which is a standard Bayesian game in the sense of Van Zandt and Vives (2007). Second, we show that a specific stochastic ordering for the family $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$ guarantees each player in $N$ prefers (in a unilateral sense) higher quality information (net of information costs), and study the unilateral effects of information on players’ strategies in the game $\Gamma_{II}(\alpha)$. Third, we address the second order effects of information acquisition on each player’s expected payoff, by postulating a natural restriction on the ordered family $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$ that leads to increasing returns in own information quality. Fourth, invoking the latter property, we then characterize the Bayesian information equilibrium of $\Gamma$. While the first two steps have antecedents in the related decision and game theoretic literatures, the last two steps are fully novel and have no related counterparts in the economics or probability/statistics literatures.

All proofs are collected in Section 6.

4.1 Monotone equilibria for the auxiliary game $\Gamma_{II}(\alpha)$

Let $\alpha \in [\underline{\alpha}, \overline{\alpha}]^n$ denote any exogenous profile of experiments. We begin with the analysis of the fictitious game $\Gamma_{II}(\alpha)$, described above.

For each player $i$ in $N$, the set of strategies $\Sigma_i$ constitutes a lattice when ordered with the pointwise (partial) order, i.e., $\sigma_i \geq \sigma'_i$ if $\sigma_i(s_i) \geq \sigma'_i(s_i)$ for every $s_i \in S_i$. We say that a strategy $\sigma_i \in \Sigma_i$ is increasing if for all $s_i, s'_i \in S_i$ such that $s_i > s'_i$, $\sigma_i(s_i) \geq \sigma_i(s'_i)$. The first result, taken from Van Zandt and Vives (2007), states that extremal equilibria for the game $\Gamma_{II}(\alpha)$ exist and are composed of strategies that are increasing in own message for each player.

**Lemma 2** For any given $\alpha \in [\underline{\alpha}, \overline{\alpha}]^n$, $\Gamma_{II}(\alpha)$ has a greatest and a least Bayesian Nash equilibrium, both of which are monotone increasing and Borel measurable in own message. Moreover, if $\Gamma$ is symmetric and $\alpha_1 = \alpha_2 = \ldots = \alpha_n$, then every equilibrium is also symmetric.

Henceforth, given a vector of information qualities $\alpha$, we select the maximal equilibrium for the game $\Gamma_{II}(\alpha)$, denoted $\overline{\alpha}$, which thus satisfies the monotonicity property of Lemma 2. This property is a critical feature of our approach, invoked to validate the procedure we use to order structures of information.\(^{17}\)

\(^{17}\)Another way to proceed here would consist of assuming the equilibrium of $\Gamma_{II}(\alpha)$ is unique as in Okuno-Fujiwara, Postlewaite and Suzumara (1990). Since complementarities often lead to multiplicity of equilibria, the latter assumption would be too restrictive for our setting.
The next step in the analysis is to investigate the conditions under which no agent \( i \) has any incentive to deviate from a given profile \( \alpha \), using Lemma 2. We first focus on the returns to information acquisition. Two key aspects deserve some attention. First, since we model hidden information acquisition, agent \( i \)'s deviation from \( \alpha_i \) to \( \alpha_i' \) will not be observable to others, so that there will be no strategic effect on the other players. The situation would be very different if we were studying a case where agents, before choosing an action in the second stage, were able to observe the profile of information acquired by the others in the first stage (i.e. the game \( \Gamma_{II} \)). Hauk and Hurkens (2001) compare the implications of these two different games for a Bayesian Cournot model with endogenous signals. Second, since player \( i \) knows the quality of his own signal, if he deviates at stage I to a given \( \alpha_i' \), he will need to accordingly adjust his strategy at stage II to continue to best respond to the fixed strategy \( \sigma_{-i}(s_{-i}) \). These two observations will be used in crucial ways, as we next investigate the value of information in the problem faced by player \( i \).

Given any profile \( \alpha \), assume players other than \( i \) follow their corresponding strategies at the (monotonic) equilibrium \( \bar{\sigma}(\alpha) \) of \( \Gamma_{II}(\alpha) \) at stage II. Define \( U_i(\alpha_i', \alpha) \) as the maximum expected payoff player \( i \) can get (net of the costs of information acquisition) if he unilaterally deviates from \( \alpha_i \) to \( \alpha_i' \) in the first stage, i.e.,

\[
U_i(\alpha_i', \alpha) \triangleq \max_{\sigma_i' \in \Sigma_i} \left\{ \int_{s \in \Omega} u_i(\sigma_i'(s_i), \bar{\sigma}_{-i}(s_{-i}), \omega) \, dF(s, \omega; \alpha_i', \alpha_{-i}) \right\}.
\]

In going from \( \alpha_i \) to \( \alpha_i' \), unobserved by others (who then still use the equilibrium strategy \( \sigma_{-i}(s_{-i}) \)), he would then switch from the strategy \( \sigma_i(\cdot) \) to some selection of

\[
\varphi_i(s_i; \alpha_i') \triangleq \arg \max_{a_i \in A_i} \int_{s_{-i}, \Omega} u_i(a_i, \bar{\sigma}_{-i}(s_{-i}), \omega) \, dF(s_{-i}, \omega \mid s_i; \alpha_i', \alpha_{-i}) .
\]

In the proof of the next result, we provide intermediate steps that show that the maximum in (10) is achieved, so \( \varphi_i(s_i; \alpha_i') \) is well-defined. While Borel measurability of \( \bar{\sigma} \) and \( U_i(\alpha_i', \alpha) \) with respect to the \( \alpha \)'s can be established (e.g. building on the approach in Van Zandt, 2010), these properties would be useful only in imparting some desirable but non-crucial regularity to the underlying outcome functions. More importantly, our main results below require placing much more structure on our primitives, which lead to \( U_i(\alpha_i', \alpha) \) being increasing and convex in \( \alpha_i' \), so Borel measurability is then a direct corollary.

**Lemma 3** The maximal and the minimal selections of \( \varphi_i(s_i; \alpha_i') \), \( \bar{\varphi}_i(s_i; \alpha_i') \) and \( \underline{\varphi}_i(s_i; \alpha_i') \), exist and are increasing (and thus Borel measurable) in \( s_i \).

Built on this monotonicity property, the next subsection compares information structures. It characterizes \( U_i(\alpha_i', \alpha) \) as a function of \( \alpha_i' \) and shows that the informativeness of a signal refers to
its association with the state of the world. It also sheds some light on the effect of \( \alpha'_i \) on players’ behavior at stage II.

### 4.2 Comparing information structures

This subsection centers on the stage of information acquisition. Our aim is to study the incentives of player \( i \) to deviate from \( \alpha_i \). We only focus on one side of his decision: the returns to the quality of information. The cost side will be incorporated into the analysis later on.

Recall \( U_i(\alpha'_i, \alpha) \) denotes agent \( i \)'s maximum expected payoff when he unilaterally deviates from \( \alpha_i \) to \( \alpha'_i \) at the start of the game. We next consider all alternative information structures \( \{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [a, \pi]} \) that he might acquire, unbeknownst to other agents, and aim to answer the following two questions: (i) What conditions will ensure that for all \( \alpha'_i > \alpha''_i \), the information structure \( F(s_i, \omega; \alpha'_i) \) is better than \( F(s_i, \omega; \alpha''_i) \) in the sense that agent \( i \) receives a higher expected payoff, i.e. \( U_i(\alpha'_i, \alpha) \geq U_i(\alpha''_i, \alpha) \)? (ii) What conditions will suffice for \( U_i(\alpha'_i, \alpha) \) to be convex (concave) in \( \alpha'_i \)? In other words, (i) aims to characterize situations where the value of covert information is positive, while (ii) addresses the issue of returns to scale in the value of information.

#### 4.2.1 First-order effects of the quality of information

The approach in this subsection is similar to the one followed by Athey and Levin (2001). To address the two posed questions, the methodology we use to order the family of signals compares structures of information that share the same marginal distributions. Although this condition about \( \{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [a, \pi]} \) seems to be very restrictive, we argue next that this is not the case, using a well known transformation.

Assume \( \tilde{\alpha}_i \) has a continuous and strictly increasing cdf.\(^{18}\) We can always convert the message that player \( i \) receives to a new message \( z_i = F(s_i; \alpha_i) \), that has the same informational content as the initial one. This so-called "probability integral transformation" makes \( \tilde{z}_i \) uniformly distributed on \([0, 1]\), so \( \tilde{z}_i \) is independent of \( \alpha_i \). Let’s define \( G(z_i, \omega; \alpha_i) = F(F^{-1}(z_i; \alpha_i), \omega; \alpha_i) \), where \( F^{-1}(z_i; \alpha_i) \) is the inverse of the marginal distribution \( F(z_i; \alpha_i) \). It can be easily verified that our new family of joint distributions \( \{G(z_i, \omega; \alpha_i)\}_{\alpha_i \in [a, \pi]} \) shares the same marginals. In addition, since in the game \( \Gamma \), signals enter payoffs only through strategies, this transformation does not affect the game in any fundamental way. Then, without loss of generality, we shall assume that our initial family \( \{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [a, \pi]} \) satisfies this (equi-marginals) property.\(^{19}\)

\(^{18}\)If the cdf of \( \tilde{\alpha}_i \) is not invertible, then a related transformation would lead to a similar result (see Angus, 2001, for the formal argument and proof).

\(^{19}\)This use of the probability integral transformation is standard in the literature (see, e.g., Athey and Levin, 2001,
To ascertain the impact of changing $\alpha_i'$ on $U_i(\alpha_i', \alpha)$ we impose more structure on the family $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$. We assume $F(s_i, \omega; \alpha_i)$ increases in $\alpha_i$ in the supermodular stochastic order. Given that this order is central to our analysis, we state a formal definition and a convenient characterization for analytical purposes.

**Definition 4** Let $F(s_i, \omega; \alpha_i)$ and $F(s_i, \omega; \alpha_i')$ denote two cdf’s that share the same marginals. Then $F(s_i, \omega; \alpha_i)$ is larger than $F(s_i, \omega; \alpha_i')$ in the supermodular order if

$$F(s_i, \omega; \alpha_i) \geq F(s_i, \omega; \alpha_i')$$

for all $(s_i, \omega) \in S_i \times \Omega$. If $F(., .; \alpha_i)$ increases in $\alpha_i$ in $[\underline{\alpha}, \overline{\alpha}]$, we say the family of information structures $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$ increases in $\alpha_i$ in the supermodular order.

The main characterization of the supermodular stochastic order refers to the expectation of supermodular functions (see Tchen, 1980, and Epstein and Tanny, 1980).

**Lemma 5** Inequality (11) holds, $\forall (s_i, \omega) \in S_i \times \Omega$, if and only if

$$\int_{S_i \times \Omega} h(s_i, \omega) dF(s_i, \omega; \alpha_i) \geq \int_{S_i \times \Omega} h(s_i, \omega) dF(s_i, \omega; \alpha_i')$$

for all supermodular functions $h(s_i, \omega)$ for which these two expectations exist.

Müller and Stoyan (2002, p. 108), Theorem 3.8.2, covers several characteristics of the supermodular order. In particular, they elaborate on the facts that the supermodular order applies to information structures that share the same marginal distributions and orders them increasingly in terms of the linear association of any increasing function of their random arguments (see also Meyer and Strulovici, 2012).

The following example offers one general way of constructing such a class of bivariate joint distributions via the weighted mixing of two fixed distributions (for use later on).

**Example 6** Let $M(s_i, \omega)$ and $N(s_i, \omega)$ denote two joint distributions that share the same marginals and satisfy $M(s_i, \omega) \geq N(s_i, \omega)$ on the entire support, i.e. $M(., .)$ is larger than $N(., .)$ in the supermodular order. In addition, assume $k : [\underline{\alpha}, \overline{\alpha}] \to [0, 1]$ is a differentiable function.

Then $F(s_i, \omega; \alpha_i) = k(\alpha_i) M(s_i, \omega) + [1 - k(\alpha_i)] N(s_i, \omega)$ is a well-defined cdf that shares the same marginals as $M(s_i, \omega)$ and $N(s_i, \omega)$. Clearly, if $k'(.) \geq 0$,

$$\frac{\partial F(s_i, \omega; \alpha_i)}{\partial \alpha_i} = k'(\alpha_i) [M(s_i, \omega) - N(s_i, \omega)] \geq 0$$

and Ganuza and Penalva, 2010).

It is remarkable to note that the relation between Definition 4 and Lemma 5 relies on the fact that, for bivariate distributions, the supermodular order is equivalent to the concordance order.
so the family \( \{ F(s_i, \omega; \alpha_i) \}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]} \) increases in \( \alpha_i \) in the supermodular order, i.e. an increase in \( \alpha_i \) results in an upward shift of the distribution function \( F(s_i, \omega; \alpha_i) \).

A little algebra shows that an increase in \( \alpha_i \) also shifts the survival function up.\(^{21}\) This is an interesting property of the supermodular order that applies only to bivariate distributions.

In terms of our set-up, this example shows that a higher \( \alpha_i \) leads to higher chances of observing high realizations of the signal when the state of the world is high, and low realizations when the state of the world is low. The Motivating Example in Section 3 offers another widely used family of bivariate distributions that fits Definition 4: The bivariate normal (see Müller, 2001). Keeping the marginal distributions of its random arguments fixed, the bivariate normal is increasing in the supermodular stochastic order if we let \( \alpha_i \) denote the correlation between the two random variables. Many other similar examples can be given.

The next proposition is the main result of this subsection. It states that if \( F(s_i, \omega; \alpha_i) \) increases in \( \alpha_i \) in the supermodular order, then player \( i \)'s maximum expected payoff of deviating from \( \alpha_i \) to \( \alpha_i' \) increases in \( \alpha_i' \), i.e. that the value of (unilateral) information is positive.

**Proposition 7** If \( \{ F(s_i, \omega; \alpha_i) \}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]} \) increases in \( \alpha_i \) in the supermodular order, then \( U_i(\alpha_i', \alpha) \) increases in \( \alpha_i' \).

The economic interpretation of Proposition 7 is simple. The strategic complementarities in the payoffs and the fact that other agents follow increasing strategies lead player \( i \) to prefer high actions when he predicts high values of \( \omega \), and low actions in the other case. When the structures of information are ordered according to the supermodular order, a higher \( \alpha_i' \) increases the association between \( \tilde{s}_i \) and the state of the world. Since the maximal selection of \( \varphi_i(s_i; \alpha_i') \), \( \overline{\varphi}_i(s_i; \alpha_i') \), increases in his message, a higher quality signal allows player \( i \) a better match between his actions, the fundamental and other players’ equilibrium strategies, thereby increasing his expected payoff.

Athey and Levin (2001) establish a similar outcome for monotone decision problems. Due to the absence of strategic interactions in their model, the signal plays a single role: it helps the agent to predict the fundamental. As Proposition 7 considers unilateral deviations in information acquisition in a model with strategic interactions, here the signal plays a dual role: it helps each player to predict the fundamental as well as the equilibrium actions of the other players. The proof of Proposition 7 extends Athey and Levin (2001) by showing that the complementarities between player \( i \)'s signal and the fundamental are preserved after integrating over his rivals’ equilibrium strategies. To this end we use some basic results from Van Zandt and Vives (2007), such as the monotonicity of equilibrium strategies in signal realizations.

\(^{21}\) The survival function of a bivariate random vector \((\tilde{x}_1, \tilde{x}_2)\) is defined as \( \mathcal{F}(a, b) \equiv \Pr(\tilde{x}_1 > a, \tilde{x}_2 > b) \).
Observe here that although the proof of Proposition 7 relies on the monotonicity property of player $i$'s greatest best reply at stage II as a function of $s_i$, stated in Lemma 3, the same result applies to any other measurable selection of his best-response correspondence. The reason is that the properties of the (maximal) value function $U_i(\alpha_i', \alpha)$ do not depend on the specific selection of the argmax under consideration.

We next shed some light on the effect of information acquisition at stage I on players' strategies at stage II. Specifically, the next proposition states that the strategy of agent $i$ becomes more spread-out when he acquires a signal that is more accurate. The intuition is easy to grasp: when the quality of his signal is higher, player $i$ places more faith in the messages he receives and this encourages him to make more extreme decisions in the second stage.

**Proposition 8** Assume the conditions of Proposition 7 are satisfied. Then $\varphi_i(s_i; \alpha_i')$ decreases in $\alpha_i$ when player $i$ receives a small message, and increases in $\alpha_i$ when the message is high.

**Remark.** This result is valid for the minimal selection of $\varphi_i(s_i; \alpha_i')$ as well.

Proposition 8 only applies to small and large realizations of player $i$’s signal. Its proof relies on the fact that when $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in \Omega}$ is ordered according to the supermodular order, the conditional distribution function $F(\omega | s_i; \alpha_i)$ increases (decreases) in $\alpha_i$ for small (large) values of $s_i$ ($\forall \omega \in \Omega$). As observed next, in specific applications, our statement results in a sharp prediction.

In the two examples of Levin (2001, pp. 665-666) the conditional cdf’s increase (decrease) in $\alpha_i$ for any message that is lower (higher) than the unconditional mean of $s_i$. In those settings, players’ strategies would rotate clockwise around the unconditional mean of $s_i$ whenever the information becomes more precise—this also happens in our Motivating Example.\(^\text{22}\)

The next subsection studies the second order effects of $\alpha_i'$ on $U_i(\alpha_i', \alpha)$. While our results on the value of information have several related antecedents in the literature on information and games/decisions, the upcoming results have no analogs at all in this literature.

### 4.2.2 Second-order effects of information quality: Convexity

We showed in the previous subsection that if $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in \Omega}$ is ordered in $\alpha_i$ according to the supermodular order, then the value of unilateral covert information is always positive. We now investigate the second order effects of information acquisition on agents' payoffs. In addition to being of substantial independent interest, the properties derived here are central to our approach to the existence of equilibria for the game $\Gamma$, and thus constitute the most important part of the present paper.

\(^{22}\)In the two examples of Levin (2001), $\theta$ plays the role of $\alpha_i$, and $x$ the role of $s_i$. 

19
To study the second order effects of changing $\alpha_i'$ on $U_i(\alpha_i', \alpha)$ we impose more structure on the family $\{ F(s_i, \omega; \alpha_i) \}_{\alpha_i \in [\alpha, \overline{\alpha}]}$. The next definition introduces the notion of convexity (concavity) in the supermodular order and the subsequent lemma characterizes the latter in terms of the expectation of supermodular functions. To the best of our knowledge, this is the first study that proposes (global) second order conditions on this class of information structures, or bivariate distributions in general. Related notions of stochastic convexity of integrals with respect to parameters of transition probabilities have been invoked in economic dynamics contexts, in Amir (1996) for first order stochastic dominance and Amir (1997) for second order stochastic dominance.

**Definition 9** Assume the family $\{ F(s_i, \omega; \alpha_i) \}_{\alpha_i \in [\alpha, \overline{\alpha}]}$ shares the same marginals. We say $F(s_i, \omega; \alpha_i)$ is convex (concave) in $\alpha_i$ in the supermodular order if

$$F(s_i, \omega | \lambda \alpha_i + (1 - \lambda) \alpha_i') \leq (\geq) \lambda F(s_i, \omega; \alpha_i) + (1 - \lambda) F(s_i, \omega; \alpha_i')$$

(13)

$\forall \alpha_i, \alpha_i' \in [\alpha, \overline{\alpha}], \forall \lambda \in [0, 1], \forall (s_i, \omega) \in S_i \times \Omega$, i.e. $F(\cdot, \cdot; \alpha_i)$ is convex (concave) in $\alpha_i$.

We next characterize this condition in terms of the expectation of supermodular functions. The following result says that the family $\{ F(s_i, \omega; \alpha_i) \}_{\alpha_i \in [\alpha, \overline{\alpha}]}$ is ordered and convex (concave) in $\alpha_i$ in the supermodular order if and only if the expectation of any supermodular function increases in $\alpha_i$ at an increasing (decreasing) rate.

**Lemma 10** Let us consider $\alpha_i, \alpha_i' \in [\alpha, \overline{\alpha}]$, and define $\alpha_i'' = \lambda \alpha_i + (1 - \lambda) \alpha_i'$. $F(s_i, \omega; \alpha_i)$ is convex (concave) in $\alpha_i$ in the supermodular order if and only if

$$\int_{S_i \times \Omega} h dF(s_i, \omega; \alpha_i'') \leq (\geq) \lambda \int_{S_i \times \Omega} h dF(s_i, \omega; \alpha_i) + (1 - \lambda) \int_{S_i \times \Omega} h dF(s_i, \omega; \alpha_i')$$

(14)

for all $\lambda \in [0, 1]$ and all supermodular functions $h(s_i, \omega)$ for which the three expectations exist.

In terms of our set-up, the supermodular order ranks informativeness in the sense that a higher $\alpha_i$ leads to higher chances of observing high realizations of the signal when the state of the world is high, and low realizations when the state of the world is low. This new notion of convexity/concavity refers to how fast a higher $\alpha_i$ raises informativeness. Convexity (concavity) means that $\alpha_i$ raises informativeness with increasing (decreasing) returns.

The next example illustrates how one can construct a family of joint distributions satisfying convexity (concavity) in the supermodular order via mixing.

**Example 11** In Example 6, in the previous subsection, we defined $F(s_i, \omega; \alpha_i) = k(\alpha_i) M(s_i, \omega) + [1 - k(\alpha_i)] N(s_i, \omega)$. Assume that $k : [\alpha, \overline{\alpha}] \rightarrow [0, 1]$ is twice differentiable.
Using Definition 9, and the fact that \( M(s_i; \omega) \geq N(s_i; \omega) \), it is easy to verify that the family \( \{ F(s_i; \omega; \alpha_i) \}_{\alpha_i \in [0, \infty]} \) is convex (concave) in \( \alpha_i \) in the supermodular order if and only if \( k''(\cdot) \geq 0 \) (\( k''(\cdot) \leq 0 \)), since

\[
\frac{\partial^2 F(s_i; \omega; \alpha_i)}{\partial \alpha_i^2} = k''(\alpha_i) [M(s_i; \omega) - N(s_i; \omega)] \geq (\leq) 0.
\]

This example points out that the conditions for convexity and concavity refer to how fast the cumulative distribution and the survival function increase in \( \alpha_i \) (assuming also that \( k'(\cdot) \geq 0 \)).

To generate a rich family of joint distributions, one can either keep the same fixed distributions \( M(s_i; \omega) \) and \( N(s_i; \omega) \) and vary the function \( k(\cdot) \), or keep the latter fixed and vary \( M(s_i; \omega) \) and \( N(s_i; \omega) \) while maintaining their ranking in the supermodular order, i.e. \( M(s_i; \omega) \geq N(s_i; \omega) \). Out of the commonly used bivariate distributions, one can easily write down many examples of parametric families that satisfy convexity in the supermodular order. Thus, in addition to having a natural economic interpretation in terms of increasing returns to information, this property is satisfied by easily generated families of distributions. Furthermore, convexity in the supermodular order enjoys several desirable properties, such as being preserved by various important operations such as convex combinations, pointwise maxima, and weak* (or other) limits.

On the other hand, as is often the case with powerful assumptions (as this is shown to be in the upcoming result), this convexity property does impose significant restrictions on the family of distributions. For instance, since convexity in the supermodular order amounts to \( F(s_i; \omega; \alpha_i) \) being convex in \( \alpha_i \), a moment’s thought will reveal that this distribution can possibly have atoms only along constant \( s_i \) or \( \omega \) lines, so that the location of these atoms would always be fixed, independently of the values of \( \alpha_i \).\(^{23}\) This is clearly a potentially significant limitation in the present context since a higher quality of information (i.e., a higher value of \( \alpha_i \)) necessarily leaves invariant those values of the signal and the fundamental (the atoms) that have a strictly positive probability. In this respect though, it is worth recalling that several related studies in information economics assume non-atomic distributions (as they posit density functions) at the outset. The absence of atoms would mitigate in an important way the restrictiveness of our new convexity assumptions.

The next result uses Lemma 10 to state that if \( \{ F(s_i; \omega; \alpha_i) \}_{\alpha_i \in [0, \infty]} \) is convex in \( \alpha_i \) in the supermodular order, then player \( i \)'s maximum expected payoff when he deviates from \( \alpha_i \) to \( \alpha_i' \) is convex in \( \alpha_i' \) as well, i.e., \( U_i(\alpha_i', \alpha) \) has increasing marginal returns in \( \alpha_i' \). Besides its independent interest as a foundation for the key property of increasing returns in the unilateral value of covert information,

\(^{23}\)In Example 11, \( F(s_i; \omega; \alpha_i) \) inherits whatever atoms \( M(s_i; \omega) \) and \( N(s_i; \omega) \) have, but the locations of these atoms are invariant to the values of \( \alpha_i \).
as mentioned earlier, this result is also the centerpiece behind the existence and characterization of the equilibria of the game $\Gamma$.

**Theorem 12** If $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in \mathbb{R}}$ is convex in $\alpha_i$ in the supermodular order, then $U_i(\alpha'_i, \alpha)$ is convex in $\alpha'_i$, for each fixed $\alpha$.

The proof of Theorem 12 relies on two key steps. Lemma 10 confirms that if the family $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in \mathbb{R}}$ is convex in $\alpha_i$ in the supermodular order, then the expected value of any supermodular function is convex in $\alpha_i$ as well. So the first part of the proof consists of showing that the problem of player $i$ can be rewritten in terms of the expectation of a function that is supermodular in $(s_i, \omega)$. The second part relies on the fact that the pointwise maximum of any collection of convex functions is convex (Rockafeller, 1970). This convexity result squares well with the robust result in Chade and Schlee (2002) that the value function for decision problems is convex in a neighborhood of zero.

### 4.2.3 Second-order effects of information quality: Concavity

In this subsection, we explore the dual question, or the scope for a player’s maximal expected payoff to be concave, as opposed to convex, in own information level. The interest in such a property is obvious, since this would in particular guarantee existence of pure strategy equilibrium via the standard topological approach.

For (one-player) decision problems under uncertainty, several studies have elaborated on the difficulties involved in getting the value of information to be globally concave, most notably Radner and Stiglitz (1984), and Chade and Schlee (2002) as noted above. We show that the present approach also sheds light on this issue.

Since Lemma 10 characterizes both the convexity and concavity of the expectation of supermodular functions, the first part of our proof of convex information value works for concavity as well. However, the second part of the proof fails as the pointwise maximum of a collection of concave functions need not be concave. While a similar approach can characterize concavity of $U_i(\alpha'_i, \alpha)$ in $\alpha'_i$, it entails a very restrictive joint concavity condition on own second-stage actions and signals, as captured by the next result.\(^{24}\)

\(^{24}\)In addition, the construction of information structures in Example 11 admits an obvious concave analog, obtained by simply reversing the relevant inequalities. In this construction, concavity of the information structure entails no more restrictiveness (in the mathematical sense) than its convexity.
Theorem 13 Assume that

\[ \int_{S_i \times \Omega} \int_{S_{-i}} u_i(a_i(s_i), \sigma_{-i}(s_{-i}), \omega) \, dF(s_{-i} | \omega; \alpha_{-i}) \, dF(s_i, \omega; \alpha'_i) \]  

(15)
is jointly concave on \( A_i \times [\alpha, \bar{\alpha}] \), i.e., in \( (a_i(s_i), \alpha'_i) \). Then \( U_i(\alpha'_i, \alpha) \) is concave in \( \alpha'_i \).

The joint concavity condition of Theorem 13 does not lend itself to a decomposition into separate components placed directly on the primitives of the game. Indeed, it can be seen by inspection that the integral (15) will be concave in \( a_i(s_i) \) if we assume that \( u_i \) is concave in \( a_i \). In addition, the integral (15) will also be concave in \( \alpha'_i \) if \( \{ F(s_i, \omega; \alpha_i) \}_{\alpha_i \in [\alpha, \bar{\alpha}]} \) is concave in \( \alpha_i \) in the supermodular order (the proof of the latter fact is a direct dual of that for the convex case). However, although the stated conditions guarantee the concavity of (15) in each of the two arguments \( a_i(s_i) \) and \( \alpha'_i \), they are not sufficient for the needed joint concavity in \( (a_i(s_i), \alpha'_i) \). Thus, the sufficient conditions for concavity are indeed more restrictive than their counterparts for convexity.

On the other hand, an important exception where concavity arises often is in economic applications such as those cited in the Introduction, with quadratic payoffs and Gaussian information structure (see Vives, 2008). In such settings, one can often obtain analytical solutions for the value of information as a function of a parameter of interest (which indexes the family of information structures), and there is usually a monotone transformation of this parameter that makes the objective function concave in information. In many cases, the latter has an intuitive economic interpretation (see, e.g., Hauk and Hurkens, 2001, and Vives, 1988). Nevertheless, the analysis of this section suggests that the resulting concavity is an artifact of this common parametrization, and that this convenient feature is of limited robustness.

On a related note, Chade and Schlee (2002, p. 433) raise an important issue discussed in our Example in Section 2: Are the increasing marginal returns in the value of information an artifact of the scale or particular units used to measure information? Is it possible to recover concavity by simply changing the units of information in a nonlinear fashion? Our analysis indicates that the scope for an affirmative answer to this question is quite limited. Indeed, by assimilating different possible information measures with the inverse cost transformation described in Section 2.2, and noting that our convexity result is robust to a whole class of valid cost functions as long as the resulting family \( \{ F(s_i, \omega; \alpha_i) \} \) satisfies the stochastic monotonicity and convexity assumptions, the aforementioned difficulties in obtaining a concave value of information would still apply. On the other hand, the cost transformation does impose a certain constraint on our procedure, in that the family \( \{ F(s_i, \omega; \alpha_i) \} \) must continue to satisfy our two key assumptions upon the application of that transformation.
4.3 Bayesian Nash equilibrium with endogenous information

This subsection combines the results in the previous two to show existence and characterize the pure strategy equilibria of $\Gamma$. Our previous analysis referred to the returns to information acquisition. Considering the cost of acquiring information, we can use Definition 1 to state that a pure strategy profile $(\alpha^*, \bar{\alpha})$ is a Bayesian Nash equilibrium with endogenous information of $\Gamma$ if

$$\alpha_i^* \in \arg \max \left\{ U_i (\alpha_i, \alpha^*) - C_i (\alpha_i) : \alpha_i \in [\bar{\alpha}, \alpha] \right\} \text{ for all } i \in N,$$

that is, if no player $i$ has an incentive to unilaterally deviate from $\alpha^*$.

**Remark 14** If $\Gamma (\alpha)$ has a unique equilibrium for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]^n$, condition (16) is also necessary.

Given any tuple $\alpha \in [\underline{\alpha}, \overline{\alpha}]^n$, we let $\alpha_i^* (\alpha)$ denote the set of argmax’s of the net payoff

$$\left\{ U_i (\alpha_i', \alpha) - C_i (\alpha_i') : \alpha_i' \in [\underline{\alpha}, \overline{\alpha}] \right\}.$$

Proposition 7 and Theorem 12 state general conditions under which $U_i (\alpha_i', \alpha)$ is increasing and convex in $\alpha_i'$. Since we normalized the cost of information to be linear in $\alpha_i'$, these conditions guarantee that the maximand in (16) is convex in $\alpha_i'$ as well. Then its argmax exists, and it is always given by the largest element of the constraint set, the smallest one, or both, as if the players were in a binary action game.

**Lemma 15** Assume $\{ F (s_i, \omega; \alpha_i) \}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$ is increasing and convex in $\alpha_i$ in the supermodular order. Then, the game with action set $[\underline{\alpha}, \overline{\alpha}]$ and payoff functions given by (16) is strategically equivalent (generically) to the $n \times 2$ game with action set $[\underline{\alpha}, \overline{\alpha}]$ and payoff functions given by (16).\(^{25}\)

The reason we used the cost transformation in Section 2.2 to generate (w.l.o.g.) a linear cost function is precisely to compound the convexity question fully on the value of information, while keeping the cost side neutral via its linearity. In other words, our assumptions of stochastic monotonicity and supermodularity are actually placed jointly on the pair $(\{ F (s_i, \omega; \alpha_i) \}, C_i)$.

If the conditions of Lemma 15 are satisfied, for any given tuple of information, all the players have incentives to deviate to either $\underline{\alpha}$ or $\overline{\alpha}$. As a consequence, any profile $\alpha$ that satisfies the equilibrium condition entails extreme behavior in the first stage. All intermediate information levels are simply strictly dominated strategies for every player. The next proposition formalizes this assertion and characterizes the possible equilibria that can arise in the entire game $\Gamma$.

\(^{25}\)"Generically" here means that $U_i (\alpha_i', \alpha) - C (\alpha_i)$ is not a constant, as otherwise one would have that every action is an optimal response. To avoid this knife-edge case, we assume this cannot happen for $\Gamma$. 

24
Proposition 16 Assume \( \{ F(s_i, \omega; \alpha_i) \} \) is increasing and convex in \( \alpha_i \) in the supermodular order. Then

(i) any equilibrium profile of experiments \( \alpha^* \) that satisfies (16) is given by \( \alpha^* \in \{ \alpha, \overline{\alpha} \}^n \);

(ii) \( \overline{\alpha} \) is an equilibrium profile of experiments if \( U_i(\overline{\alpha}, \overline{\alpha}) - U_i(\alpha, \overline{\alpha}) \geq \alpha - \alpha, \forall i \in N \); and

(iii) \( \alpha \) is an equilibrium profile of experiments if \( U_i(\alpha, \alpha) - U_i(\overline{\alpha}, \overline{\alpha}) \geq \alpha - \alpha, \forall i \in N \).

Proposition 16 characterizes the pure strategy equilibria of \( \Gamma \), taking into account that the second-stage equilibrium strategies are given by \( \overline{\sigma}(\alpha^*) \), which is the maximal equilibrium actions in the sense of Van Zandt and Vives (2007), given information profile \( \alpha^* \). We now make a number of remarks concerning the scope of Proposition 16.

Proposition 16 does not rule out hybrid equilibria wherein a subset of players use strategy \( \alpha \) while the remaining players use strategy \( \overline{\alpha} \). Conditions for such hybrid equilibria similar to those given in Proposition 16 (ii)-(iii) can obviously be written down. An example of such an equilibrium configuration is given below.

Conditions (ii) and (iii) of Proposition 16 are not mutually exclusive; both \( \overline{\alpha} \) and \( \alpha \) are equilibria in settings that satisfy

\[
U_i(\overline{\alpha}, \overline{\alpha}) - U_i(\alpha, \overline{\alpha}) \geq \overline{\alpha} - \alpha \geq U_i(\overline{\alpha}, \overline{\alpha}) - U_i(\alpha, \alpha)
\]

(17)

for all \( i \in N \). Inequality (17) requires the supermodularity of \( U_i(\alpha', \overline{\alpha}) \) in \( (\alpha', \overline{\alpha}) \) at the extreme values of these profiles. In words, player \( i \)'s incremental payoff of deviating from \( \alpha \) to \( \overline{\alpha} \) must be higher when the profile of information is \( \overline{\alpha} \) than when it is \( \alpha \).

Elaborating on the previous remark, since it is well-known that every \( 2 \times 2 \) game is either a supermodular game or a matching-pennies game (Echenique, 2004), the game at hand for the case of two players is also of one of these two types.\(^\text{26}\) It follows that the game defined by (16) has a pure-strategy equilibrium if and only if the \( 2 \times 2 \) game with action space \( \{ \alpha, \overline{\alpha} \} \) is supermodular with respect to one of the four possible ways of ordering the two binary action sets in the spirit of Echenique (2004).\(^\text{27}\) In this sense, our approach to ranking information structures leads to existence of pure-strategy equilibrium in the two-stage game by exploiting strategic complementarities in both stages of the game, with the important caveat that these complementarities are required only for the two extreme levels of information. Although complementarities do emerge for all levels of

\(^{26}\)As its name suggests, a matching-pennies game is a \( 2 \times 2 \) game with a unique mixed-strategy Nash equilibrium, which is thus globally stable for a large class of learning dynamics.

\(^{27}\)We cannot in general extend this statement to the game \( \Gamma \) itself, since that would entail taking into account all possible increasing equilibrium selections in the second stage of the game.
information acquisition in models with linear-quadratic payoffs and Gaussian information, the scope for extending this powerful property to general formulations appears rather limited. Thus the present relaxation of this requirement, that it holds only for the two extreme levels of information acquisition, appears promising for economic applications with general functional forms.

If inequality (17) is satisfied for all \( i \) in \( N \), then \( \overline{\alpha} \) is Pareto-preferred to \( \alpha \) if (and only if) \( U_i (\overline{\alpha}, \overline{\alpha}) - U_i (\alpha, \alpha) \geq \overline{\alpha} - \alpha \) for all \( i \) in \( N \). It can be easily shown that the last condition is always satisfied if, as in the Motivating Example, \( U_i (\alpha', \alpha) \) increases in \( \alpha \).

Thinking of \( \overline{\alpha} \) as the full information signal, i.e. a signal that reveals the true value of Nature with certainty, then Proposition 16 (ii) states sufficient conditions for the complete information game to emerge as an endogenous outcome. For the sake of tractability in much applied work based on game-theoretic analysis, we often assume that all the economic fundamentals are common knowledge. Our result helps to identify environments where this assumption fits better the underlying phenomenon.

We end the section with a brief discussion about the costs of information. Arrow (1974) points out that as data observation often calls for an initial amount of resources that is independent of the scale of operation, then production of information often has increasing returns to scale. Radner (2000) adds that the situation is even more delicate in the case of electronic communications, as the marginal cost of new information is far smaller than the initial cost of development. Although our normalization of the cost function does not capture fixed costs, their introduction would not change the bang-bang nature of our results in any qualitative way. A fixed cost would only alter results in favor of the uninformative outcome occurring more readily at equilibrium.

In cases where the players in the game \( \Gamma \) purchase the desired information from some specialized suppliers, it is plausible to postulate an actual cost schedule faced by the players that is concave in the level of information, as a reflection of a pricing policy based on volume discounts. This would provide another avenue for the overall payoff function to be strictly convex in own information level, thus reinforcing the qualitative nature of our results.

### 4.4 On the possibility of asymmetric extreme equilibria

As noted earlier, Proposition 16 encompasses the possibility of quite asymmetric information equilibria. Specifically, we could observe a situation where, at equilibrium, some of the players opt for the highest quality signal, while their rivals decide to remain uninformed, and this even in games that are themselves symmetric. We next illustrate such a situation with a closed-form duopoly example, but one that does not satisfy all the maintained assumptions of the game \( \Gamma \).

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28This is the case for the oligopoly settings mentioned in the Introduction as well as other settings, such as Hellwig and Veldkamp (2009), in addition to the specific examples given in the present paper.
Example 17 Consider a market with two ex ante identical firms that compete in quantities, where firm i’s action is \( q_i \geq 0 \). Assuming linear inverse demand \( p_i = \omega - q_i/2 - \delta q_{-i} \), and constant unit costs normalized to zero, firm i’s profits are given by

\[
\pi_i = \omega q_i - q_i^2/2 - \delta q_{-i} q_i, \quad i = 1, 2
\]

(18)

where \( \delta \in (0, 1) \) and \( \omega \) is the realization of a random parameter drawn from a normal prior with mean \( \mu \) and variance \( \sigma^2_\omega \). The remaining features of the model, including the structures of information, are analogous to the Motivating Example. As a consequence, we can follow the same steps to conclude that a profile of information \( \alpha \) is an equilibrium if it satisfies, simultaneously, the next system

\[
\begin{align*}
\alpha_1 &\in \arg \max_{\alpha'_1 \in [0,1]} \left\{ \frac{1}{2} \left[ \left( \frac{\mu}{1+\delta} \right)^2 + v_\omega^2 \left( \frac{1 - \delta \alpha_2^2}{1 - (\delta \alpha_1 \alpha_2)^2} \right)^2 \alpha_1^2 \right] - \alpha'_1 \right\} \\
\alpha_2 &\in \arg \max_{\alpha'_2 \in [0,1]} \left\{ \frac{1}{2} \left[ \left( \frac{\mu}{1+\delta} \right)^2 + v_\omega^2 \left( \frac{1 - \delta \alpha_1^2}{1 - (\delta \alpha_2 \alpha_1)^2} \right)^2 \alpha_2^2 \right] - \alpha'_2 \right\} .
\end{align*}
\]

(19)

Since the maximands of (19) are strictly convex in \( \alpha'_i \in [0,1] \), their argmax’s are either 0 or 1, for \( i = 1, 2 \). Therefore, any equilibrium entails extreme behavior regarding information acquisition.

It can be easily verified that in this game \( \alpha = (1, 0) \) and \( \alpha = (0, 1) \) constitute an information-equilibria pair if and only if \( v_\omega^2 \geq 2 \geq v_\omega^2/(1+\delta)^2 \).

Although the firms are ex-ante identical and share the same prior information, their posteriors are maximally different at equilibrium. Moreover, if the firms differ in their marginal costs of information, the outcome where the firm with higher costs gets the full-information signal and the other one remains uninformed, is an equilibrium (for some parameter values).

This study focuses on common-value Bayesian supermodular games in which players’ second-stage actions are strategic complements. Although this duopoly example satisfies instead the property of strategic substitutes, it is well known that this game can be turned into a supermodular Bayesian game by reversing the order on the strategy space of one of the players. However, upon this order reversal, the game would no longer have both profit functions increasing in \( \omega \).

A remark on the connection between the present framework and the theory of common knowledge in the sense of Aumann (1976) is in order here.\(^{29}\) In this example of strategic interaction with endogenous information, the two players start with a common prior (as is standard for Bayesian games), but end up with maximally differentiated posteriors at the unique equilibrium of the game. This outcome is consistent with the conclusions from the theory of common knowledge since, by the very nature of the game here, the two divergent posteriors are not common knowledge between the

\[^{29}\text{A thorough survey of the relevance of this literature in economic settings is Samuelson (2004).}\]
two players. Nevertheless, this feature of the example may be of interest to the interface between game theory and common knowledge (Samuelson, 2004).

5 Concluding remarks

The paper studies endogenous information acquisition in common-value Bayesian supermodular games. Since there is a pronounced tendency towards nonconcavities in the value of information for decision problems, as shown by Radner and Stiglitz (1984) and Chade and Schlee (2002) and confirmed by the present results, a more general approach to existence of pure-strategy Bayesian equilibrium with endogenous signals through standard topological fixed point theorems appears infeasible. A similar remark applies to the supermodularity approach and Tarski’s fixed point theorem, as the complementarities of the second stage are, in general, not inherited by the information decisions of the first stage, at least at the level of generality postulated in the present research. On the other hand, in various economic applications that postulate linear-quadratic payoffs and Gaussian information, two important properties often emerge that our analysis suggests are nonrobust. The first is that expected payoffs are concave in own information and the second is that the first-stage information acquisition decisions tend to be strategic complements in a global sense (Vives, 2008).

Our approach to the equilibrium existence problem relies on the supermodular order to rank information structures, which allows for a convenient way to introduce a notion of stochastic convexity that captures quite naturally the property of increasing returns in information acquisition, and operationalizes it as a basis for a drastic simplification of the overall game and the equilibrium existence issue. The key step is that each player’s payoff is strictly convex in own information level, so that interior strategies are always strictly dominated choices.

While our results on the value of information acquisition have several related antecedents in the literature on information and games/decisions, the second-order properties developed here have no analogs in game-theoretic settings. From an economic perspective, the increasing returns property leads to extreme behavior regarding information acquisition. As such, this analysis may justify the emergence in different settings of the complete information game, of a standard Bayesian game or both at the same time, all as endogenous outcomes. As shown via example, the framework can also deliver maximally asymmetric information equilibria even with ex ante identical players.

Along the way, we also elaborate on the meaning of information in the class of games we focus on: We show that if the structures of information are increasing in the supermodular stochastic order, then players’ incremental returns to information acquisition (in a unilateral sense) are always
positive. Then, informativeness relates to a particular form of association between the signal and the unknown. As we mentioned earlier, this result has related antecedents. Regarding the effects of information acquisition on players’ second decisions, we find that the informativeness of the signals leads to actions that are more spread-out. The intuition is easy to grasp: When the quality of his signal is higher player \( i \) places more faith in the messages he receives, and this encourages him to make more extreme decisions later on.

6 Proofs

Proof of Lemma 2: For any given \( \alpha \in [\underline{\alpha}, \overline{\alpha}]^n \), our Initial Assumptions guarantee that \( \Gamma_{II} (\alpha) \) is a monotone supermodular Bayesian game as defined by Van Zandt and Vives (2007). Then existence of maximal and minimal Bayesian Nash equilibria, in strategies that are increasing in own signal, follows directly from their main result on p. 344. The symmetry statement is on p. 346.

Most of our proofs require the usual characterization of first order stochastic dominance.

Lemma 18 Let \( \vec{x} \) and \( \vec{x}' \) denote two random vectors with support \( X \subseteq \mathbb{R}^n \) and respective distribution functions \( F(\cdot) \) and \( F'(\cdot) \). For any \( n \geq 1 \), \( \vec{x} \) first order stochastically dominates \( \vec{x}' \) if

\[
\int_X h(x) \, dF(x) \geq \int_X h(x) \, dF'(x)
\]

for all bounded increasing measurable functions \( h: \mathbb{R}^n \rightarrow \mathbb{R} \).

Although the next proof follows from Propositions 8 and 11 in Van Zandt and Vives (2007), we provide it for completeness.

Proof of Lemma 3: Assume players other than \( i \) acquire profile \( \alpha_{-i} \) at stage I, and follow \( \sigma_i (s_i) \) at stage II. If player \( i \) deviates from \( \alpha_i \) to \( \alpha_i' \) in the first stage, recall the definitions

\[
U_i (\alpha_i', \alpha) \triangleq \max_{\alpha'_i \in \Sigma_i} \left\{ \int_{S_i} \int_{\Omega} u_i (\sigma_i' (s_i) , \sigma_{-i} (s_{-i}) , \omega) \, dF (s_i , \omega ; \alpha_i' , \alpha_{-i}) \right\}
\]

and

\[
\varphi_i (s_i; \alpha_i') \triangleq \arg\max_{\alpha_i \in A_i} \int_{S_i \times \Omega} u_i (\alpha_i , \sigma_{-i} (s_{-i}) , \omega) \, dF (s_{-i} , \omega \mid s_i; \alpha_i' , \alpha_{-i}) .
\]

We next study the properties of the maximal selection of \( \varphi_i (s_i; \alpha_i') \), denoted \( \varphi_i (s_i; \alpha_i') \).

Continuity and supermodularity are preserved by integration. Therefore, by Assumptions 1 and 2, for all \( s_i \in S_i \), the maximand of (21) is continuous and supermodular in \( a_i \). It follows from Lemma 7 in Van Zandt and Vives (2007, p. 348) that, for all \( s_i \in S_i \), \( \varphi_i (s_i; \alpha_i') \) is a non-empty complete sublattice, and that \( \varphi_i (s_i; \alpha_i') = \sup \varphi_i (s_i ; \alpha_i') \) belongs to \( \varphi_i (s_i; \alpha_i') \). The fact that \( \varphi_i (s_i; \alpha_i') \) is measurable in \( s_i \), so that \( \varphi_i (s_i; \alpha_i') \) belongs to \( \Sigma_i \), is shown in Van Zandt (2010).
We next show the maximand of (21) has increasing differences in \((a_i, s_i)\), given that \(\mathcal{F}(s_{-i})\) is increasing. Let \(a_i \geq a'_i\) and \(s_i \geq s'_i\), and consider the following steps:

\[
\begin{align*}
&\int_{s_{-i}} u_i (a_i, \mathcal{F}(s_{-i}), \omega) \, dF(s_{-i}, \omega | s_i; \alpha'_{i}, \alpha_{-i}) \\
&\quad - \int_{s_{-i}} u_i (a'_i, \mathcal{F}(s_{-i}), \omega) \, dF(s_{-i}, \omega | s_i; \alpha'_{i}, \alpha_{-i}) \\
&= \int_{s_{-i}} [u_i (a_i, \mathcal{F}(s_{-i}), \omega) - u_i (a'_i, \mathcal{F}(s_{-i}), \omega)] \, dF(s_{-i}, \omega | s_i; \alpha'_{i}, \alpha_{-i}) \\
&\geq \int_{s_{-i}} [u_i (a_i, \mathcal{F}(s_{-i}), \omega) - u_i (a'_i, \mathcal{F}(s_{-i}), \omega)] \, dF(s_{-i}, \omega | s'_i; \alpha'_{i}, \alpha_{-i}) \\
&= \int_{s_{-i}} u_i (a_i, \mathcal{F}(s_{-i}), \omega) \, dF(s_{-i}, \omega | s'_i; \alpha'_{i}, \alpha_{-i}) \\
&\quad - \int_{s_{-i}} u_i (a'_i, \mathcal{F}(s_{-i}), \omega) \, dF(s_{-i}, \omega | s'_i; \alpha'_{i}, \alpha_{-i}).
\end{align*}
\]

By Lemma 2, \(\mathcal{F}(s_{-i})\) is increasing and, by Assumption 2, \(u_i\) has increasing differences in \((a_i; a_{-i}, \omega)\). Then 

\[
u_i (a_i, \mathcal{F}(s_{-i}), \omega) - u_i (a'_i, \mathcal{F}(s_{-i}), \omega)
\]

increases in both \(s_{-i}\) and \(\omega\). In addition, by Assumption 3, \(F(s_{-i}, \omega | s_i; \alpha'_{i}, \alpha_{-i})\) first order stochastically dominates the distribution \(F(s_{-i}, \omega | s'_i; \alpha'_{i}, \alpha_{-i})\). So the inequality follows by Lemma 18. This shows the maximand of (21) has increasing differences in \((a_i, s_i)\). Hence \(\bar{\nu}(s_i; \alpha'_i)\) is increasing in \(s_i\). A similar argument applies to \(\inf \phi (s_i; \alpha'_i)\).

**Proof of Proposition 7:** The proof of this proposition consists of two steps. Step 1 shows that for any \(\sigma_i \in \Sigma_i\) that is increasing, \(\int_{s_{-i}} u_i (\sigma_i(s_i), \mathcal{F}(s_{-i}), \omega) \, dF(s_{-i} | \omega; \alpha_{-i})\) has increasing differences in \((s_i, \omega)\). Step 2 uses this to show that \(U_i (\alpha'_i, \alpha)\) increases in \(\alpha'_i\).

**Step 1.** Assume \(\sigma_i \in \Sigma_i\) is increasing, \(s_i \geq s'_i\) and \(\omega \geq \omega'\), and consider the next inequalities

\[
\begin{align*}
&\int_{s_{-i}} u_i (\sigma_i(s_i), \mathcal{F}(s_{-i}), \omega) \, dF(s_{-i} | \omega; \alpha_{-i}) \\
&\quad - \int_{s_{-i}} u_i (\sigma_i(s'_i), \mathcal{F}(s_{-i}), \omega) \, dF(s_{-i} | \omega; \alpha_{-i}) \\
&= \int_{s_{-i}} [u_i (\sigma_i(s_i), \mathcal{F}(s_{-i}), \omega) - u_i (\sigma_i(s'_i), \mathcal{F}(s_{-i}), \omega)] \, dF(s_{-i} | \omega; \alpha_{-i}) \\
&\geq \int_{s_{-i}} [u_i (\sigma_i(s_i), \mathcal{F}(s_{-i}), \omega) - u_i (\sigma_i(s'_i), \mathcal{F}(s_{-i}), \omega)] \, dF(s_{-i} | \omega'; \alpha_{-i}) \\
&\geq \int_{s_{-i}} [u_i (\sigma_i(s_i), \mathcal{F}(s_{-i}), \omega') - u_i (\sigma_i(s'_i), \mathcal{F}(s_{-i}), \omega')] \, dF(s_{-i} | \omega'; \alpha_{-i}) \\
&= \int_{s_{-i}} u_i (\sigma_i(s_i), \mathcal{F}(s_{-i}), \omega') \, dF(s_{-i} | \omega'; \alpha_{-i}) \\
&\quad - \int_{s_{-i}} u_i (\sigma_i(s'_i), \mathcal{F}(s_{-i}), \omega') \, dF(s_{-i} | \omega'; \alpha_{-i}).
\end{align*}
\]

First, since \(\mathcal{F}(s_{-i})\) is increasing, \(u_i (\sigma_i(s_i), \mathcal{F}(s_{-i}), \omega) - u_i (\sigma_i(s'_i), \mathcal{F}(s_{-i}), \omega)\) increases in \(s_{-i}\) by
Assumption 2. In addition, by Assumption 3, \( F(s | \omega; \alpha) \) increases in \( \omega \) in the sense of first order stochastic dominance. Then, by property MA of Theorem 3.3.10 in Müller and Stoyan (2002, p. 94), \( F(s_{-i} | \omega; \alpha_{-i}) \) increases in \( \omega \) according to the same order. As a consequence, the first inequality follows by Lemma 18. By Assumption 2, \( u_i(\sigma_i, \overline{\sigma}_i(s_{-i}), \omega) - u_i(\sigma_i(s_i'), \overline{\sigma}_i(s_{-i}), \omega) \) increases in \( \omega \), so this difference is lower at \( \omega' \) than at \( \omega \), for all \( s_{-i} \in S_{-i} \): this justifies the second inequality.

This argument shows that \( \int_{s_{-i}} u_i(\sigma_i(s_i), \overline{\sigma}_i(s_{-i}), \omega) dF(s_{-i} | \omega; \alpha_{-i}) \) has increasing differences in \( (s_i, \omega) \), i.e., Step 1. We now use this result to show that \( U_i(\alpha_i', \alpha) \) increases in \( \alpha_i' \).

**Step 2.** Let \( \varphi_i(s_i; \alpha_i') \) be the maximal selection of \( \varphi_i(s_i; \alpha_i') \), defined in (21) and \( \alpha_i' > \alpha_i'' \). Then,

\[
U_i(\alpha_i', \alpha) = \int_{S_i} \int_{S_{-i}} u_i(\varphi_i(s_i; \alpha_i'), \overline{\sigma}_i(s_{-i}), \omega) \, dF(s_{-i} | \omega; \alpha_{-i}) \, dF(s_i, \omega; \alpha_i')
\]

\[
\geq \int_{S_i} \int_{S_{-i}} u_i(\varphi_i(s_i; \alpha_i''), \overline{\sigma}_i(s_{-i}), \omega) \, dF(s_{-i} | \omega; \alpha_{-i}) \, dF(s_i, \omega; \alpha_i')
\]

\[
\geq \int_{S_i} \int_{S_{-i}} u_i(\varphi_i(s_i; \alpha_i''), \overline{\sigma}_i(s_{-i}), \omega) \, dF(s_{-i} | \omega; \alpha_{-i}) \, dF(s_i, \omega; \alpha_i'')
\]

\[
= U_i(\alpha_i'', \alpha)
\]

The two equalities are true by definition and because players’ signals are assumed independent given the state of the world, i.e. \( F(s_{-i} | s_i, \omega; \alpha_{-i}) = F(s_{-i} | \omega; \alpha_{-i}) \). The first inequality follows by the optimality principle. We know, by Lemma 3, that \( \varphi_i(s_i; \alpha_i') \) increases in \( s_i \). Then Step 1 guarantees the integral

\[
\int_{S_{-i}} u_i(\varphi_i(s_i; \alpha_i'), \overline{\sigma}_i(s_{-i}), \omega) \, dF(s_{-i} | \omega; \alpha_{-i})
\]

is supermodular in \( (s_i, \omega) \). Since \( \{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]} \) increases in \( \alpha_i \) in the supermodular order, the second inequality follows by Lemma 5.

**Proof of Proposition 8:** The cdf \( F(\omega, s_i; \alpha_i) \) and the survival function \( \overline{F}(\omega, s_i; \alpha_i) \) can be rewritten

\[
F(\omega, s_i; \alpha_i) = \int_{-\infty}^{s_i} F(\omega | t; \alpha_i) \, dF(t)
\]

\[
\overline{F}(\omega, s_i; \alpha_i) = 1 - F(s_i) - \int_{s_i}^{\infty} F(\omega | t; \alpha_i) \, dF(t).
\]

where \( F(.) \) stands for the marginal cdf of player \( i \)'s message, and \( F(\omega | t; \alpha_i) \) represents the conditional cdf of the fundamental given a message \( s_i = t \). Let \( \alpha_i, \alpha_i' \in [\underline{\alpha}, \overline{\alpha}] \) satisfy \( \alpha_i > \alpha_i' \). Since we
assumed the conditions of Proposition 7, the next (in)equalities hold for all \( s_i \in S_i \) and \( \omega \in \Omega \)

\[
\int_{-\infty}^{s_i} \left[ F(\omega | t; \alpha_i) - F(\omega | t; \alpha'_i) \right] dF(t) \geq 0
\]
\[
\int_{s_i}^{\infty} \left[ F(\omega | t; \alpha_i) - F(\omega | t; \alpha'_i) \right] dF(t) \leq 0
\]
\[
\int_{-\infty}^{\infty} \left[ F(\omega | t; \alpha_i) - F(\omega | t; \alpha'_i) \right] dF(t) = H(\omega) - H(\omega) = 0.
\]

The second inequality holds because for bivariate distributions the supermodular stochastic order coincides with the concordance order (see Example 6 and the footnote 20).

Therefore, for any fixed \( \omega \in \Omega \) there exists a value \( s_\omega \in S_i \) such that for all \( s_i \leq s_\omega \) we have \( F(\omega | s_i; \alpha_i) \geq F(\omega | s_i; \alpha'_i) \). Take \( \underline{s} = \inf \{ s_\omega : \omega \in \Omega \} \). By construction, if \( s_i \leq \underline{s} \) then \( F(\omega | s_i; \alpha_i) \geq F(\omega | s_i; \alpha'_i) \) for all \( \omega \in \Omega \). Using a similar argument, we can find a value \( \overline{s} \) such that \( F(\omega | s_i; \alpha_i) \leq F(\omega | s_i; \alpha'_i) \) for all \( \omega \in \Omega \).

According to Lemma 3 and Assumption 2, \( \overline{\sigma}_{-i} \) is increasing, \( u_i \) is supermodular in \( \alpha_i \) and it has increasing differences in \( (a_i; a_{-i}, \omega) \). In addition, by Assumption 3, \( F(s | \omega; \alpha) \) increases in \( \omega \) in the first order stochastic dominance. Then, by Müller and Stoyan (2002, p. 94), property MA of Theorem 3.3.10, \( F(s_{-i} | \omega; \alpha_{-i}) \) increases in \( \omega \) according to the same order. Therefore we can apply Lemma 18 to conclude that the next function increases in \( \omega \) for all \( \alpha_i \geq \alpha'_i \)

\[
\int_{S_{-i}} \left[ u_i(a_i, \overline{\sigma}_{-i}(s_{-i}), \omega) - u_i(a'_i, \overline{\sigma}_{-i}(s_{-i}), \omega) \right] dF(s_{-i} | \omega; \alpha_{-i}).
\]

Applying Lemma 18 again we get that

\[
\int_{\Omega} \int_{S_{-i}} u_i(a_i, \overline{\sigma}_{-i}(s_{-i}), \omega) dF(s_{-i} | \omega; \alpha_{-i}) dF(\omega | s_i; \alpha_i)
\]

\[
- \int_{\Omega} \int_{S_{-i}} u_i(a'_i, \overline{\sigma}_{-i}(s_{-i}), \omega) dF(s_{-i} | \omega; \alpha_{-i}) dF(\omega | s_i; \alpha_i)
\]

increases (decreases) whenever \( s_i \leq \underline{s} \) (\( s_i \geq \overline{s} \)) and player \( i \) chooses \( \alpha'_i \) instead of \( \alpha_i \). Then the incremental returns with respect to own action of the maximand in \( (21) \) are higher at \( \alpha'_i \) than at \( \alpha_i \) when the message is small (high). Since the latter is supermodular in \( \alpha_i \), Proposition 8 follows from Lemma 7 in Van Zandt and Vives (2007, p. 348).

Proof of Lemma 10: Since \( \lambda F(s_i, \omega; \alpha_i) + (1 - \lambda) F(s_i, \omega; \alpha'_i) \) is a convex combination of two cdf’s that share the same marginals, it is also a cdf with the same marginal distributions \( F(s_i) \) and \( H(\omega) \), respectively.

Since \( F(s_i, \omega; \lambda \alpha_i + (1 - \lambda) \alpha'_i) \) and \( \lambda F(s_i, \omega; \alpha_i) + (1 - \lambda) F(s_i, \omega; \alpha'_i) \) have the same marginal distributions, we can try to compare them in terms of the supermodular stochastic order. Let \( \alpha''_i = \lambda \alpha_i + (1 - \lambda) \alpha'_i \). Assuming \( h(s_i, \omega) \) is a supermodular function with finite expectation with respect to both cdf’s, we know, by Lemma 5, that the following two conditions are equivalent
(i) \( F(s_i, \omega; \alpha''_i) \leq (\geq) \lambda F(s_i, \omega; \alpha_i) + (1 - \lambda) F(s_i, \omega; \alpha'_i) \) \( \forall (s_i, \omega) \in S_i \times \Omega \)

(ii) \( \int_{S_i \times \Omega} h dF(s_i, \omega; \alpha''_i) \leq (\geq) \int_{S_i \times \Omega} h d [\lambda F(s_i, \omega; \alpha_i) + (1 - \lambda) F(s_i, \omega; \alpha'_i)] \).

Since expectation is a linear operator, condition (ii) is in turn equivalent to

(iii) \( \int_{S_i \times \Omega} h dF(s_i, \omega; \alpha''_i) \leq (\geq) \lambda \int_{S_i \times \Omega} h dF(s_i, \omega; \alpha_i) + (1 - \lambda) \int_{S_i \times \Omega} h dF(s_i, \omega; \alpha'_i) \).

Thus (i) is satisfied if and only if (iii) is fulfilled, which is exactly our claim.

**Proof of Theorem 12:** Let \( \lambda \in [0, 1] \), \( \alpha'_i \) and \( \alpha''_i \) denote two arbitrary elements of \([\underline{\alpha}, \overline{\alpha}]\), and \( \alpha''_i = \lambda \alpha'_i + (1 - \lambda) \alpha''_i \). The following inequalities show our statement,

\[
\begin{align*}
\lambda U_i (\alpha'_i, \alpha) + (1 - \lambda) U_i (\alpha''_i, \alpha) \\
= \lambda \int_{S_i \times \Omega} \int_{S_{-i}} u_i (\bar{\varphi}_i (s_i; \alpha'_i), \bar{\sigma}_{-i} (s_{-i}), \omega) \, dF (s_{-i} | \omega; \alpha_{-i}) \, dF (s_i, \omega; \alpha'_i) \\
+ (1 - \lambda) \int_{S_i \times \Omega} \int_{S_{-i}} u_i (\bar{\varphi}_i (s_i; \alpha''_i), \bar{\sigma}_{-i} (s_{-i}), \omega) \, dF (s_{-i} | \omega; \alpha_{-i}) \, dF (s_i, \omega; \alpha''_i) \\
\geq \lambda \int_{S_i \times \Omega} \int_{S_{-i}} u_i (\bar{\varphi}_i (s_i; \alpha''_i), \bar{\sigma}_{-i} (s_{-i}), \omega) \, dF (s_{-i} | \omega; \alpha_{-i}) \, dF (s_i, \omega; \alpha'_i) \\
+ (1 - \lambda) \int_{S_i \times \Omega} \int_{S_{-i}} u_i (\bar{\varphi}_i (s_i; \alpha''_i), \bar{\sigma}_{-i} (s_{-i}), \omega) \, dF (s_{-i} | \omega; \alpha_{-i}) \, dF (s_i, \omega; \alpha''_i) \\
\geq \int_{S_i \times \Omega} \int_{S_{-i}} u_i (\bar{\varphi}_i (s_i; \alpha''_i), \bar{\sigma}_{-i} (s_{-i}), \omega) \, dF (s_{-i} | \omega; \alpha_{-i}) \, dF (s_i, \omega; \alpha''_i) \\
= U_i (\alpha''_i, \alpha)
\end{align*}
\]

where the first inequality is true since \( \bar{\varphi}_i (s_i; \cdot) \) is a best-response to \( \bar{\sigma}_{-i} (s_{-i}) \), and the second one follows from the following argument. Consider the function

\[
\int_{S_{-i}} u_i (\bar{\varphi}_i (s_i; \alpha''_i), \bar{\sigma}_{-i} (s_{-i}), \omega) \, dF (s_{-i} | \omega; \alpha_{-i}) .
\]

Lemma 2 ensures \( \bar{\varphi}_i (s_i; \alpha''_i) \) is increasing in \( s_i \), so we can use Step 1 in the proof of Proposition 7 to confirm that (22) is supermodular in \((s_i, \omega)\). Then the second inequality follows by Lemma 10, since \( \{ F(s_i, \omega; \alpha_{i}) \}_{\alpha_{i} \in [\underline{\alpha}, \overline{\alpha}]} \) is convex in \( \alpha_i \) in the supermodular stochastic order.

**Proof of Theorem 13:** Let \( \lambda \in [0, 1] \), \( \alpha'_i \) and \( \alpha''_i \) denote two arbitrary elements of \([\underline{\alpha}, \overline{\alpha}]\), and
\[ \alpha_i'' = \lambda \alpha_i' + (1 - \lambda) \alpha_i'' . \]
The following inequalities prove the result,

\[
U_i (\alpha_i'', \alpha) = \int_{S_i \times \Omega} \int_{s_{-i}} u_i (\varphi_i (s_i; \alpha_i''), \sigma_{-i} (s_{-i}), \omega) \ dF (s_{-i} | \omega; \alpha_{-i}) \ dF (s_i, \omega; \alpha_i'') \\
\geq \int_{S_i \times \Omega} \int_{s_{-i}} \left[ u_i (\lambda \varphi_i (s_i; \alpha_i') + (1 - \lambda) \varphi_i (s_i; \alpha_i''), \sigma_{-i} (s_{-i}), \omega) \right] \ dF (s_{-i} | \omega; \alpha_{-i}) \ dF (s_i, \omega; \alpha_i') \\
+ (1 - \lambda) \int_{S_i \times \Omega} \int_{s_{-i}} u_i (\varphi_i (s_i; \alpha_i''), \sigma_{-i} (s_{-i}), \omega) \ dF (s_{-i} | \omega; \alpha_{-i}) \ dF (s_i, \omega; \alpha_i'') \\
= \lambda U_i (\alpha_i', \alpha) + (1 - \lambda) U_i (\alpha_i'', \alpha)
\]
where the first inequality follows from the optimality of \( \varphi_i \) and the second from the assumption of joint concavity.

**Proof of Lemma 15:** By Proposition 7 and Theorem 12, our conditions imply \( U_i (\alpha_i', \alpha) - C_i (\alpha_i') \) is convex in \( \alpha_i' \) for all \( \alpha \in [\underline{\alpha}, \bar{\alpha}]^n \) and all \( i \) in \( N \), given that \( -C_i (\alpha_i) = - (\alpha_i - \underline{\alpha}) \) is linear (and thus convex) and the sum of convex functions is a convex function. It follows from this convexity property that \( U_i (\alpha_i', \alpha) - C_i (\alpha_i') \) is continuous in \( \alpha_i' \) for all \( \alpha \in [\underline{\alpha}, \bar{\alpha}]^n \) and upper semi-continuous at \( \alpha_i' = \underline{\alpha} \) and \( \alpha_i' = \bar{\alpha} \). Hence, \( U_i (\alpha_i', \alpha) - C_i (\alpha_i') \) achieves its maximum in \( \alpha_i' \) for each \( \alpha \in [\underline{\alpha}, \bar{\alpha}]^n \).

The same convexity property also implies that, irrespective of the initial profile \( \alpha \in [\underline{\alpha}, \bar{\alpha}]^n \), each player’s best reply will always be in \( \{ \underline{\alpha}, \bar{\alpha} \} \), i.e., one of the two extreme qualities at stage I, for generic games. Hence, the game is strategically equivalent to the game with action set \( \{ \underline{\alpha}, \bar{\alpha} \} \) and payoffs as given by (16), but restricted to these binary action sets.

**Proof of Proposition 16**

Part (i) is a direct consequence of Lemma 15. Parts (ii) and (iii) follow as a simple application of Lemma 15 via the relevant comparison of the binary payoffs for each player.
References


Dynamics and Control*, 22, 67-86.

36, 652-654.


