Counting lines

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For the purposes of the talk, all varieties are assumed to be over $\mathbb{C}$.

The main subject of this talk is Schubert Calculus. Schubert calculus can be productively thought of as an analog to Bezout's theorem for Grassmanians, (And isn't $\mathbb{P}^n$ just the grassmanian of lines in a vector space?), so I will start with a discussion of Bezout's Theorem.

1 Bezout's Theorem and Homology of Projective Space

Bezout's Theorem is a classic theorem in Algebraic Geometry. It says the following

Theorem 1.1. *Let X and Y be subvarieties of $\mathbb{P}^n$ such that $\dim X + \dim Y = n$, and X and Y meet transversely. Then X and Y meet in a finite number of points, and $\#(X \cap Y) = \deg X \cdot \deg Y$*

Let X and Y be subvarieties of $\mathbb{P}^n$ such X and Y meet transversely. Then $\deg(X \cap Y) = \deg X \cdot \deg Y$

By meet transversely, I mean that at every point in the intersection both X and Y are smooth, and at each point p in the intersection $T_pX + T_pY \simeq T_p\mathbb{P}^n$.

Remark 1.2. Depending on how you define degree, this is almost a tautology. Also, this theorem is amenable to generalization, you can drop the transversality requirement if you are willing to be scheme-y

I want to re-state this in a method that will generalize nicely to the the case of grassmanians. For any variety X, define $Z(X)$, the group of cycles on X, as the free abelian group generated by the irreducible subvarieties of X. Note two things:

- $Z(X)$ has a natural grading coming from dimension. I will refer to the subgroup of dimension i subvarieties as $Z_i(X)$
- I can talk without confusion about reducible subvarieties as also being contained in $Z(X)$ by writing them as sums

$\mathbb{P}^n$ is naturally a smooth manifold, so we also have the groups $H_*(\mathbb{P}^n, \mathbb{Z})$, the singular homology groups with coefficients in $\mathbb{Z}$. For what follows I will usually omit the $\mathbb{Z}$. Recall that $H_j(\mathbb{P}^n) = \mathbb{Z}$ if and only if k is even and $0 \leq k \leq 2n$, and the groups are 0 for all other k. Furthermore, $H_0(\mathbb{P}^n)$ is canonically isomorphic to $\mathbb{Z}$.

Also, since $\mathbb{P}^n$ is smooth and compact, we are in a situation where Poincare Duality applies, so $H_*(\mathbb{P}^n)$ gets a ring structure from the (dual of the) cup product. I will abuse notation and write the product as $\alpha \smile \beta$.

The restatement/corrolary of Bezout's theorem is as follows:

Theorem 1.3. *There is a surjection, called the* cycle class map $cl_{\mathbb{P}^n} : Z(\mathbb{P}^n) \to H_*(\mathbb{P}^n)$, *such that when X and Y are subvarieties of $\mathbb{P}^n$ meeting transversely, $cl_{\mathbb{P}^n}(X \cap Y) = cl_{\mathbb{P}^n}(X) \smile cl_{\mathbb{P}^n}(Y)$. Moreover, for any hyperplane $\mathbb{P}^k \subset \mathbb{P}^n$, $cl_{\mathbb{P}^n}(\mathbb{P}^k)$ does not depend on the specific hyperplane and generates the group $H_{2k}(\mathbb{P}^n)$.*

Proof. This proof goes by induction. The case when $n = 0$ is clear, $\mathbb{P}^0$ is just a point, so we send the class of the point in $Z(\mathbb{P}^0)$ to the canonical generator of $H_0(\{*\})$.

Now assume the theorem is true for $n-1$. Pick a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. The inclusion induces the vertical maps in the diagram.

$$\begin{array}{ccc} Z(\mathbb{P}^{n-1}) & \xrightarrow{cl_{\mathbb{P}^{n-1}}} & H_*(\mathbb{P}^{n-1}) \\ \downarrow & & \downarrow \\ Z(\mathbb{P}^n) & \dashrightarrow & H_*(\mathbb{P}^{n-1}) \end{array}$$

Since $\mathbb{P}^n$ comes from attaching a (real) $2n$-cell to $\mathbb{P}^{n-1}$, the vertical map on the right is an isomorphism in grading $2n-1$ and lower. We can use this to define $cl_{\mathbb{P}^n}$ on the image of $Z(\mathbb{P}^{n-1})$ inside $Z(\mathbb{P}^n)$. This also shows that the class of $\mathbb{P}^{n-1}$ generates $H_{2n-2}(\mathbb{P}^n)$, and we induct to see this for all $\mathbb{P}^k$ with $k < n$.

We can also define the map on $Z_n(\mathbb{P}^n) = \mathbb{Z}[\mathbb{P}^n]$ by sending the cycle $\mathbb{P}^n$ to the canonical generator of $H_n(\mathbb{P}^n)$.

For two different hyperplanes P_1 and P_2, we know that the class of either generates $H_{2n-2}(\mathbb{P}^n)$, but they could differ by a sign. This would be bad, but luckily it doesn't happen:

$GL_{n+1}(\mathbb{C})$ acts on $\mathbb{P}^n$, and some g takes P_1 to P_2. Taking a path from Id to g in $GL_{n+1}(\mathbb{C})$ defines a homotopy between the map two maps $\mathbb{P}^{n-1} \to \mathbb{P}^n$ coming from identifying $\mathbb{P}^{n-1}$ with P_1 and P_2 respectively, so the maps in homology are the same.

Now we have a map out of the subgroup of $Z(\mathbb{P}^n)$ spanned by the elements that are contained in some hyperplane. We can extend this to a map from all of $Z(\mathbb{P}^n)$. Given any irreducible subvariety $X \subset \mathbb{P}^n$ of dimension k, we know $cl_{\mathbb{P}^n}(X)$ should be $d \cdot cl_{\mathbb{P}^n}(\mathbb{P}^k)$ for some integer d. Finding an $n-k$-plane that meets X transversely, we see that $d = \deg X$.

Now we are done. If X and Y are any two subvarieties that meet transversely, we can find a hyperplane of complementary dimension to $X \cap Y$ which is simultaneously transverse to X and Y, and intersecting with this hyperplane and thinking about Bezout's theorem tells us exactly that $cl_{\mathbb{P}^n}(X \cap Y) = cl_{\mathbb{P}^n}(X) \smile cl_{\mathbb{P}^n}(Y)$. □

This isn't very new, but it highlights a key point. We can use the decomposition $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$ to understand the homology of $\mathbb{P}^n$ in terms of the homology of $\mathbb{P}^{n-1}$.

2 The cycle map and homology for $\mathbb{G}(1,3)$

The goal of this talk is to answer questions like the following:

Given 4 lines in $\mathbb{P}^3$, with no two coplanar, how many lines are there meeting all 4 lines?

Bezout's theorem can be interpreted (if you squint) as a theorem about counting points. We do this by studying the space that parametrizes the points in $\mathbb{P}^n$ (also known as $\mathbb{P}^n$). So, it stands to reason, that if we want to know about lines in $\mathbb{P}^3$ we should study $\mathbb{G}(1,3)$. Recall the grassmanian is a smooth, variety of dimension 4.

We want, just like in the case of $\mathbb{P}^n$ to understand the homology of the grassmanian by giving it a CW-structure with only even dimensional (real) cells. The good news is we can do that!

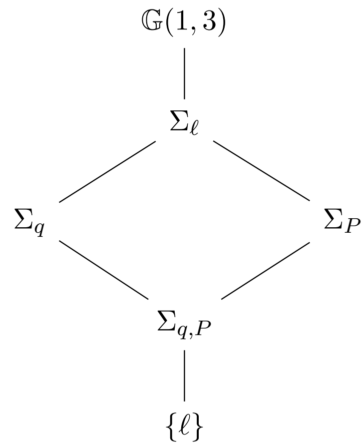
Fix the following

- q is a point in $\mathbb{P}^3$
- ℓ is a line in $\mathbb{P}^3$ containing q
- P is a plane in $\mathbb{P}^3$ containing ℓ.

Then we have the following closed subvarieties in $\mathbb{G}(1,3)$, called the *Schubert Cycles*

- Σ_ℓ the subvariety of all lines meeting ℓ
- Σ_q, the subvariety of lines containing q
- Σ_P the subvariety of lines contained in P
- $\Sigma_{q,P}$ the subvariety of lines containing q and contained in P
- $\{\ell\}$, the point in $\mathbb{G}(1,3)$ corresponding to the line ℓ.

It is very helpful to arrange these subvarieties according to dimension and containment



The theorem we wish to prove is that these classes once again freely generate the structure of homology

Theorem 2.1. *There is an surjection* $cl_{\mathbb{G}(1,3)} : Z(\mathbb{G}(1,3)) \to H_*(\mathbb{G}(1,3))$, *such that the classes of the Schubert Cycles in the above diagram freely generate* $H_*(\mathbb{G}(1,3))$ *as an additive group. Furthermore, the classes of the schubert cycles are unchanged if you vary* q, ℓ *or* P.

Moreover, whenever X *and* Y *are subvarieties of* $\mathbb{G}(1,3)$ *meeting transversely,* $cl_{\mathbb{G}(1,3)}(X \cap Y) = cl_{\mathbb{G}(1,3)}(X) \smile cl_{\mathbb{G}(1,3)}(Y)$.

Proof. First, we prove the additive structure. Analogous to the case for $\mathbb{P}^n$ this will follow from the following claims.

- $\mathbb{G}(1,3)$ comes from attaching an even dimensional (real) cell to Σ_ℓ
- Σ_ℓ comes from attaching an even dimensional (real) cell to $\Sigma_P \cup \Sigma_q$
- $\Sigma_P \cup \Sigma_q$ comes from attaching two even dimensional (real) cells to $\Sigma_{q,P}$
- $\Sigma_{q,P}$ is isomorphic to $\mathbb{P}^1$

We will address the claims in reverse order. The fourth claim is obvious.

For the third claim, note first that $\Sigma_P \cap \Sigma_q = \Sigma_{q,P}$. Then observe that Σ_P and Σ_q are both isomorphic to $\mathbb{P}^2$, so their union is obtained from $\Sigma_{q,P}$ by attaching $\mathbb{A}^2$ twice. The argument that $Z_2(\mathbb{G}(1,3))$ is isomorphic to the homology of a union of two projective planes meeting in a projective line is slightly more compilcated, but follows easily from Mayer-Vietoris.

Pick a plane H containing p but not ℓ. Then we get a map

$$\Sigma_\ell \backslash \Sigma_P \cup \Sigma_q \to (\ell \backslash \{q\}) \times (H \backslash (P \cap H)) \simeq \mathbb{A}^1 \times \mathbb{A}^2$$

This map sends a line ℓ' to $(\ell' \cap \ell, \ell' \cap H)$. It is not hard to see it is an isomorphism.

Finally, choosing H_1 and H_2 to be two planes containing ℓ we get a map

$$\mathbb{G}(1,3)\backslash\Sigma_\ell \to (H_1\backslash\ell) \times (H_2\backslash\ell) \simeq \mathbb{A}^2 \cap \mathbb{A}^2$$

By sending a line ℓ' to $(\ell' \cap H_1, \ell' \cap H_2)$.

This shows that the classes of the schubert cycles generate homology additively. We can make a similar argument to before, that the class of a schubert cycle does not depend on the choice of q, ℓ or P. Here is the argument for Σ_ℓ, the other cases are identical: Pick two lines ℓ and ℓ'. The group $GL_4(\mathbb{C})$ acts on $\mathbb{G}(1,3)$ and contains an element g that brings the class Σ_ℓ onto $\Sigma_{\ell'}$. This shows the two subvarieties are isomorphic, and a path from Id to g defines a homotopy and shows the equivalence of the classes in homology.

Now we need to show that the multiplicative structure captures transverse intersections, and can be extended to all subvarieties of $\mathbb{G}(1,3)$.

We need the following claim

Lemma 2.2. *Let X be any subvariety in $\mathbb{G}(1,3)$, then there exist some q, ℓ and P such that the corresponding Schubert Cycles are transverse to X. In fact, for a general choice of q, ℓ , and P, the schubert cycles are transverse*

Proof. Letting X be any subvariety of $\mathbb{G}(1,3)$ and C any schubert cycle. Saying C and X intersect transversely is the same as saying that $C \cap X$[1] is smooth, and of the "expected" dimension. Consider the variety

$$\Gamma = \{(c, x, g) \subset C \times X \times GL_4(\mathbb{C}) | g \cdot c = x\}$$

$GL_4(\mathbb{C})$ acts transitively on $\mathbb{G}(1,3)$, so the projection $\Gamma \to C \times X$ is surjectve. The fiber of a point (c, x) is the stabilizer of c in $GL_4(\mathbb{C})$, so the fiber has dimension 3, so the dimension of Γ is $\dim(C) + \dim(X) + 3$.

On the other hand we have a projection map $\Gamma \to GL_4(\mathbb{C})$. The fiber over a point g is isomorphic to $gC \cap X$. Moreover the general fiber is of dimension $\dim(C) + \dim(X) - 4$, which is the correct "expected" dimension of the intersection.

Finally, since we are over characteristic 0, the map $(\Gamma\backslash\Gamma_{sing}) \to G$ is smooth over an open subset of G, so for the genreal fiber is smooth and of the correct dimension. □

Now, we can see what $cl_{\mathbb{G}(1,3)}(X) \smile cl_{\mathbb{G}(1,3)}(Y)$ *should be* for X and Y Schubert cycles, by making two different choices for $q \subset \ell \subset P$ and $q' \subset \ell' \subset P'$ and counting lines in the intersection. This turns out to be tractable! Here are the computations.

Note that we can canonically identify $H_0(\mathbb{G}(1,3))$ with $\mathbb{Z}$, and we can also have a canonical map, the *degree map* $\deg : Z_0(X) \to \mathbb{Z}$, that sends a zero cycle $\sum n_i[q_i] \to \sum n_i$. The degree map factors through the cycle class map. Because of this, I will frequently refer to elements of $H_0(\mathbb{G}(1,3))$ as simply an integer.

[1] This needs to be the scheme theoretic intersection

- $cl_{\mathbb{G}(1,3)}(\Sigma_q)^2 = cl_{\mathbb{G}(1,3)}(\Sigma_P)^2$ should be 1, since there is exactly one line containing in 2 points, or contained in 1 plane

- $cl_{\mathbb{G}(1,3)}(\Sigma_q) \smile cl_{\mathbb{G}(1,3)}(\Sigma_P)$ should be 0, since for a general point q and plane P' there there are no lines that simultaneously contain q and are contained in P', since q will not be contained in P'.

- $cl_{\mathbb{G}(1,3)}(\Sigma_\ell) \smile cl_{\mathbb{G}(1,3)}(q, P)$ should be 1. Given a general point q, a general plane P containing q, and a general line ℓ', there is only one line containing q, contianed in P and meeting ℓ'.

- $cl_{\mathbb{G}(1,3)}(\Sigma_\ell)^2$ is an element of $H^2(\mathbb{G}(1,3))$ so it can be written as $\alpha cl_{\mathbb{G}(1,3)}(\Sigma_q) + \beta cl_{\mathbb{G}(1,3)}(\Sigma_P)$ for some integers α and β. Thinking about $cl_{\mathbb{G}(1,3)}(\Sigma_\ell)^2 \smile cl_{\mathbb{G}(1,3)}(\Sigma_q)$ and $cl_{\mathbb{G}(1,3)}(\Sigma_\ell)^2 \smile cl_{\mathbb{G}(1,3)}(\Sigma_P)$ shows that α and β should both be 1

- $cl_{\mathbb{G}(1,3)}(\Sigma_\ell) \smile cl_{\mathbb{G}(1,3)}(\Sigma_q)$ and $cl_{\mathbb{G}(1,3)}(\Sigma_\ell) \smile cl_{\mathbb{G}(1,3)}(\Sigma_P)$ should both be $cl_{\mathbb{G}}(1,3)(\Sigma_{q,P})$ by the same reasoning as in the previous point.

Now (and this is the part where you're going to have to take my word for it), the homology ring for $\mathbb{G}(1,3)$ turns out to look exactly like this! That is, it has one generator in dimensions 0, 2, 6 and 8, two generators in dimension 4, and the generators satisfy exactly the multiplicative properties outlined above.

It remains to extend the map to every subvariety in $\mathbb{G}(1,3)$. If X is an irreducible subvariety of dimension d, then by intersecting X with transverse schubert cycles in *codimension* d, and counting points, we can determine uniquely what the class of X should be. This will make it so that when X and Y are transverse, their intersection is precisely the product of their classes. □

3 Actual Computations

It's been a while since we stated it, but we can now answer the question posed earlier. The number of lines meeting four general lines, can be determined by computing

$$cl_{\mathbb{G}(1,3)}(\Sigma_\ell)^4 = (cl_{\mathbb{G}(1,3)}(\Sigma_q) + cl_{\mathbb{G}(1,3)}(\Sigma_P))^2 = cl_{\mathbb{G}(1,3)}(\Sigma_q)^2 + cl_{\mathbb{G}(1,3)}(\Sigma_P)^2 = 2$$

This seems like a lot of work for a small result (Indeed, a result you could prove classically, if you first prove the result that three general lines in $\mathbb{P}^3$ are contained in a unique quadric surface). But it trivializes a great deal of problems concerning lines in $\mathbb{P}^n$, and generalizes to describe a structure on $Z(\mathbb{G}(k,n))$ for other grassmanians. Here are some questions you are now equipped to answer with little difficulty:

- If C_1, C_2, C_3 and C_4 are four general curves of degree $d_1, ..d_4$, how many lines meet all 4 curves?

- If C_1 and C_2 are two general twisted cubics, how many lines are simultaneously chords of both C_1 and C_2?
- If S_1, S_2, S_3 and S_4 are surfaces in $\mathbb{P}^3$ of degree $d_1 - d_4$, how many lines are simultaneously tangent to all 4?

There is not a chance that I get here in a talk, but this construction is also used to prove there are 27 lines on a smooth cubic surface, though it takes significantly more work to realize the cycle "Lines on a smooth cubic surface" as an intersection of other subvarieties in the grassmanian.