Examples of the Brauer-Manin Obstruction

October 13, 2022

1 Review

For k a field, and X a scheme over k, we have defined $\mathrm{Br}(X)$, a contravariant functor from schemes to abelian groups. Using this contravariance, for any element A in $\mathrm{Br}(X)$ we get a map $\mathrm{ev}_A : X(k) \to \mathrm{Br}(k)$[1] by taking the image of A under the map $\mathrm{Br}(X) \to \mathrm{Br}(k)$ coming from applying the functor Br to the inclusion $\mathrm{Spec}(k) \hookrightarrow X$.

When k is a number field, we write $\mathbb{A}$ for the adelic ring of k and this extends to a commutative diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbb{A}) \\ \downarrow{\scriptstyle \mathrm{ev}_A} & & \downarrow{\scriptstyle \mathrm{ev}_A} \\ \mathrm{Br}(k) & \longrightarrow & \oplus_v \mathrm{Br}(k_v) \end{array}$$

Here, the right vertical map is the sum of the maps $\mathrm{ev}_A : X(k_v) \to \mathrm{Br}(k_v)$.

This is useful to determining k points, because there for all v there are maps $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, which can be summed to get a map $\mathrm{inv} : \oplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ making the bottom row exact.

$$\begin{array}{ccccccccc} & & X(k) & \longrightarrow & X(\mathbb{A}) & & & & \\ & & \downarrow{\scriptstyle \mathrm{ev}_A} & & \downarrow{\scriptstyle \mathrm{ev}_A} & & & & \\ 0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \oplus_v \mathrm{Br}(k_v) & \xrightarrow{inv} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

So for any A in $\mathrm{Br}(X)$ any point in $X(\mathbb{A})$ could only lie in the image of $X(k)$ if it lies in the kernel of $\mathrm{inv} \circ \mathrm{ev}_A$.

We define $X(\mathbb{A})^{\mathrm{Br}} := \{x \in X(\mathbb{A}) | \mathrm{inv} \circ \mathrm{ev}_A(x) = 0 \text{ for all } A \in \mathrm{Br}(X)\}$, so clearly $X(k) \subset X(\mathbb{A})^{\mathrm{Br}}$

The goal of this talk will be to show examples and non-examples of this obstruction.

[1] I will abuse notation and write $\mathrm{Br}(k)$ instead of $\mathrm{Br}(\mathrm{Spec}(k))$

2 Brauer-Severi Varieties and Projective Quadrics

Following [2] Chapter 7 A Severi-Brauer Variety over a field k is a variety X such that there exists a field extension K/k with $X \times_k K \simeq \mathbb{P}^n_K$

As twists of $\mathbb{P}^n$, Brauer-Severi varieties are classified up to k-isomorphism by elements of $H^1(\Gamma_k, PGL_n(k^s))$ There is a short exact sequence of Γ_k modules

$$1 \longrightarrow (k^s)^* \longrightarrow GL_n(k^s) \longrightarrow PGL_n(k^s) \longrightarrow 1$$

Considering the corresponding long exact sequence in group cohomology, and recalling Hilbert's Theorem 90 (i.e. $H^1(\Gamma_k, (k^s)^*) = 0$) we see that there is an injection

$$H^1(\Gamma_k, PGL_n(k^s)) \hookrightarrow H^2(\Gamma_k, (k^s)^*) \simeq \mathrm{Br}(k)$$

So to every Brauer-Severi variety X we can associate a class $A_k(X)$ in the Brauer group of k.

Châtelet's Theorem states that for any Severi-Braeur Variety X over a field k, $X(k) \neq \emptyset$ if and only if $X \simeq \mathbb{P}^n_k$ if and only if $A_k(X) = 0$.

This is (mostly) sufficient to see that the local-to-global principle holds for Brauer-Severi varieties. We need to know as well that the map A is compatible with base change. That is, if K is an extension of k

$$A_K(X \times_k K) = Res_{K/k} A_k(X)$$

Once we know this, the proof is as follows. Let X be a Brauer-Severi variety over a number field k. At a place v of k, if $X \times k_v$ has a k_v-point, then $A_{k_v}(X \times k_v) = 0$, by Chatelet. If this is true for all v, then $A_k(X)$ is in the kernel of the map

$$\mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v)$$

But this map is an injection, so $A_k(X) = 0$ and thus X has a k point.

With this established, let's look at our example, a much weaker result

Proposition 1. *Brauer-Severi varieties have no Brauer-Manin obstruction to the local-global principle*

We will show this by (partially) computing $\mathrm{Br}(Y)$ for Y any Brauer-Severi variety, and then showing that for all Brauer-Severi Varieties X over a number field k, The Brauer-Manin set of X is $\prod_v X(k_v)$.

For any scheme X over a field k, with structure morphism $\pi : X \to \mathrm{Spec}(k)$, the Leray Spectral sequence has E^2 page

$$H^p(\Gamma_k, H^q_{et}(X^s, \mathbb{G}_m)) \Rightarrow H^{p+q}_{et}(X, \mathbb{G}_m)$$

The (longer) short exact sequence of low degree terms gives

$$\begin{aligned}0 &\to H^1(\Gamma_k, \Gamma(X^s, \mathcal{O}^*_{X^s})) \to Pic(X) \to Pic(X)^{\Gamma_k} \\ &\to H^2(\Gamma_k, \Gamma(X^s, \mathcal{O}^*_{X^s})) \to Br_1(X) \to H^1(\Gamma_k, Pic(X^s))\end{aligned}$$

Here, $Br_1(X)$ is, following the notation used in Coliot-Thelene and Skorobogatov, $\mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X^s)]$.

For X^s connected and projective, the global sections of $\mathcal{O}_{X^s}$ are constant functions. We can once more invoke Hilbert's Theorem 90, as well as recall that $\mathrm{Br}(\mathbb{P}^n_{k^s}) = 0$ to reduce our sequence to

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X)^{\Gamma_k} \to \mathrm{Br}(k) \to \mathrm{Br}(X) \to H^1(\Gamma_k, \mathrm{Pic}(X^s))$$

For X a Brauer-Severi variety, $X^s \simeq \mathbb{P}^n$ and $\mathrm{Pic}(X^s) \simeq \mathbb{Z}$ is generated by the class of a hyperplane. $\mathrm{Pic}(X^s)$ is invariant under the action of Γ^k. This means the last term in the sequence is $\mathrm{Hom}_{cont}(\Gamma_k, \mathbb{Z}) = 0$ and we have $\mathrm{Br}(k)$ surjects on to $\mathrm{Br}(X)$.

It turns out, that the kernel of this map is the cyclic subgroup generated by $A_k(X)$, but for our purposes, surjection is sufficient.

Now returning to our general diagram for the Brauer-Manin obstruction, we see that any choice of an element α in $\mathrm{Br}(X)$ must lift to some element in $\widetilde{\alpha}$ in $\mathrm{Br}(k)$, so evaluating at α must give a *constant* map $X(k) \to \mathrm{Br}(k)$, and therefore the Brauer-Manin set is the entirety of $\prod X(k_v)$

We see the same phenomenon when Q is a smooth quadric. Once again, the local-to-global principle for quadric surfaces is established by the Hasse-Minkowski theorem, and we can use surjectivity of the map $\mathrm{Br}(k) \to \mathrm{Br}(Q)$ to establish by identical reasoning that there is no Brauer-Manin obstruction.

The same sequence above used for Q instead of X is very similar.

$$\begin{aligned}0 &\to H^1(\Gamma_k, \Gamma(Q^s, \mathcal{O}^*_{Q^s})) \to \mathrm{Pic}(Q) \to \mathrm{Pic}(Q)^{\Gamma_k} \\ &\to H^2(\Gamma_k, \Gamma(Q^s, \mathcal{O}^*_{Q^s})) \to Br_1(Q) \to H^1(\Gamma_k, \mathrm{Pic}(X^s)\end{aligned}$$

Since Q^s is projective and connected, we can use Hilbert's Theorem 90 once more, and since X^s is birational to $\mathbb{P}^n$, and the Brauer group is a birational invariant, $\mathrm{Br}(X^s) = 0$, so we once more have

$$0 \to \mathrm{Pic}(Q) \to \mathrm{Pic}(Q)^{\Gamma_k} \to \mathrm{Br}(k) \to \mathrm{Br}(Q) \to H^1(\Gamma_k, \mathrm{Pic}(Q^s))$$

If $\mathrm{Dim}(Q) = 1$ then Q is a Brauer-Severi variety and we have already seen that $\mathrm{Br}(k)$ surjects onto $\mathrm{Br}(Q)$. In this case it is also not hard to see that the kernel is the class $A(Q)$ in $\mathrm{Br}(k)$.

$A(Q)$ is the kernel of the map $\mathrm{Br}(k) \to \mathrm{Br}(K(Q))$, which factors through the injection $\mathrm{Br}(Q) \to \mathrm{Br}(K(Q)$, so $A(Q)$ lies in the kernel. Meanwhile, the map $\mathrm{Pic}(Q) \to \mathrm{Pic}(Q)^{\Gamma_k}$ contains $2\mathbb{Z}$ in its image, by considering the image of the canonical class of Q in $\mathrm{Pic}(Q)$.

When $n = \mathrm{Dim}(\mathbb{Q}) > 2$ then $\mathrm{Pic}(Q^s) \simeq \mathrm{Pic}(\mathbb{P}^n)$ so $H^1(\Gamma_k, \mathrm{Pic}(Q^s))$ is trivial, and in fact we have an isomorphism $\mathrm{Br}(k) \simeq \mathrm{Br}(Q)$

When $n = 2$ $\mathrm{Pic}(Q^s) \simeq \mathbb{Z} \times \mathbb{Z}$ generated by the two rulings of the quadric surface. The action of Γ_k is either trivial, or permutes these two rulings, and in either case $H^1(\Gamma_k, \mathrm{Pic}(Q^s)) = 0$. The same argument before concerning the surjectivity of $\mathrm{Br}(k)$ onto $\mathrm{Br}(Q)$

3 Iskovskikh's conic bundle

Following [1] Chapter 8.2

A conic over a field is the zero locus in $\mathbb{P}^2$ of a nonzero degree 2 homogenous polynomial. Another way of saying this is that it is a nonzero element of $\mathrm{Sym}^2(k^3)$.

To generalize to a variety X, take a rank 3 vector bundle $\mathcal{E}$ with a nowhere vanishing section $s \in \Gamma(X, \mathrm{Sym}^2 \mathcal{E})$ The zero locus of s in $\mathrm{Proj}(\mathrm{Sym}\,\mathcal{E})$ is called *a conic bundle* on X. Note that the fibers over a closed point are all conics in the sense above.

A *Châtelet Surface* is a further specialization above. We take

- $X = \mathbb{P}^1_k$ for $\mathrm{char}(k) \neq 2$, defined in the variables x and w
- $\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)$
- $s = (1, a, -F(x, w))$ for $a \in k^*$ and $F(x, w) \in \Gamma(\mathbb{P}^1, \mathcal{O}(4))$ a separable degree 4 homogenous polynomial.

The result is a (generically?) smooth, projective, geometrically integral surface S. It contains as an open the affine surface

$$y^2 - az^2 = f(x)$$

Where $f(x) = F(x, 1)$.

S comes with a map $S \to \mathbb{P}^1$, and the fibers of this map are smooth conics except where F is zero, in which case they are (geometrically) two intersecting lines.

Iskovsikh's example is the surface S where $k = \mathbb{Q}$, $a = 1$ and $f(x) = (3 - x^2)(x^2 - 2)$.

Proposition 2. *S has a Brauer-Manin obstruction to having rational points*

Proof. First note that S has points everywhere locally. I'll refrain from handling the places of bad reduction. To handle finite places where S is smooth, note that every integer n is a sum of two squares mod p for any p. At the real place, if $2 < x^2 < 3$ then there are come y and z to find a real point.

Now we need to choose a Brauer class on S. Set $K = k(S)$ By purity, an element of $\mathrm{Br}(K)$ that has no residues at any divisor restricts to an element of $Br(S)$. That is to say, we pick an element A of $\mathrm{Br}(K)$ and open sets U_i that cover S such that A extends to an element of $\mathrm{Br}(U_i)$ for all i.

Define $\alpha = (3 - x^2, -1)$, $\beta = (x^2 - 2, -1)$ and $\gamma = (3/x^2 - 1, -1)$.

Recall that, in the Brauer group $(a, b) + (c, b) = (ac, b)$. Then

$$\alpha + \beta = (y^2 + z^2, -1)$$

This is a split algebra because y^2+z^2 is the norm of $y+\sqrt{-1}z$ in $K(\sqrt{-1})$. Since quaternion algebras are order 2, this proves $\alpha = \beta$ in $\mathrm{Br}(K)$.

Similarly, $\alpha - \gamma = (1, -1)$ which is trivial.

α is defined everywhere away from the zeroes and poles of $3 - x^2$. β is defined on an open set containing the zeroes, γ is defined on an open set containing the poles, so we have a well defined element of $\mathrm{Br}(S)$, which we will call A.

Now we need to understand $\mathrm{inv}_v(A(P))$ for $P \in S(\mathbb{Q}_v)$ for all places v.

Case I Suppose $v = p \notin \{2, \infty\}$. set $x = x(P) \in \mathbb{Q}_p \cup \infty$. If $x = \infty$ or $v_p(x) < 0$ then $3/x^2 - 1 \in \mathbb{Z}_p^\times$. Otherwise either $3 - x^2$ or $x^2 - 2$ is in $\mathbb{Z}^\times$ since the two expressions sum to 1.

In any case, we can write $A = (u_1, u_2)$ for $u_i \in \mathbb{Z}_p^\times$. Since $p \neq 2$, this means A extends a Azumaya algebra over $\mathbb{Z}_p$. But we know $\mathrm{Br}(\mathbb{Z}_p) = 0$, so $A(P) = 0$

Case II For $v = \infty$ $S(\mathbb{R})$ contains no points with $x(P) = \infty$, since $-x^4$ cannot be a sum of squares. Either $3 - x^2$ or $x^2 - 2$ must be positive, so A can be written as $(n, -1)$ with $n > 0$, so A is trivial.

Case III For $v = 2$ then

$$v(x) > 0 \Rightarrow 3 - x^2 \equiv -1 \pmod 4$$
$$v(x) = 0 \Rightarrow x^2 - 2 \equiv -1 \pmod 4$$
$$v(x) < 0 \Rightarrow 3/x^2 - 1 \equiv -1 \pmod 4$$

But for $a \equiv -1 \bmod 4$, a is not a norm of $\mathbb{Q}_2[\sqrt{-1}]$, so in this case A is not trivial.

We then have that for any point s of $S(\mathbb{A})$, $\mathrm{inv}(A(s) = \frac{1}{2}$. Therefore $S(\mathbb{A})^{Br} = \emptyset$. □

Apparently, Iskovskikh constructed this example to demonstrate a case where BMO could not be used to show a lack of rational points, and only later was it demonstrated that in fact the Brauer-Manin set is empty in this case.

It is a result of Colliot-Thélène, Sansuc, and Swinnerton-Dyer that for Chatelet surfaces, the Brauer-Manin obstruction is the only obstruction to the Hasse-Principle, and furthermore $S(k)$ is dense in $S(\mathbb{A})^{Br}$.

This has been extended to conic bundles over $\mathbb{P}^1$ with 5 singular fibers, and is conjectured to hold for any number of singular fibers.

4 Weak Approximation

A variety X satisfies *weak* approximation if $X(k)$ is dense in $X(\mathbb{A})$. Since $X(k) \subset X(\mathbb{A})^{Br}$ and $X(\mathbb{A})^{Br}$ is closed in $X(\mathbb{A})$, then whenever $X(\mathbb{A})^{Br} \neq X(\mathbb{A})$ we know immediately that X does not satisfy weak approximation.

Proposition 3. *Let X be a scheme over a number field k such that there is an element $A \in \mathrm{Br}(X)$ and a place w of k such that the map $\mathrm{ev}_A : X(k_w) \to \mathrm{Br}(k_w)$ is non-constant. Then X does not satisfy weak approximation*

Proof. If X does not have a k-point it does not satisfy weak approximation, so assume $P \in X(k)$ is a k-point. Then the image of P under the inclusion $X(k) \hookrightarrow X(\mathbb{A})$ is an adelic point, which we will denote $(P_v)_v$

By assumption, there is a point Q in $X(k_w)$ such that $\mathrm{ev}_A(Q) \neq \mathrm{ev}_A(P_w)$. Then we can construct a new adelic point $(P'_v)_v$ such that for $v \neq w$ $P'_v = P_v$ and $P'_w = Q$. Then

$$inv\,\mathrm{ev}_A(P') = inv_w(\mathrm{ev}_A(Q)) - inv_w(\mathrm{ev}_A(P_w)) \neq 0$$

Therefore $X(\mathbb{A})^{Br} \neq X(\mathbb{A})$ and weak approximation fails. □

In fact, if X is proper and there is a finite set S of places v where A is not identically 0 on $X(k_v)$. $X(k)$ is not even dense in $\prod_S X(k_v)$.

This example is due to Swinnerton-Dyer. Let U be the affine surface

$$y^2 + z^2 = (4x - 7)(x^2 - 2) \neq 0$$

This is another conic bundle over $\mathbb{P}^1$. Take X to be its compactification. A similar argument to the eariler example shows that $A = (4x - 7, -1)$ gives a well defined class in $\mathrm{Br}(X)$. See [2] 6.3.1 for a more general principle for the construction of some quaternion algebras on conic bundles of this form.

We can similarly argue that for p not 2 or ∞, $\mathrm{ev}_A : X(\mathbb{Q}_p) \to \mathrm{Br}(\mathbb{Q}_p)$ is the zero map. In this case however, the map is zero on $X(\mathbb{Q}_2)$ as well.

$U(\mathbb{R})$ has two connected components, corresponding to $-\sqrt{2} < x < \sqrt{2}$ and $x > 7/4$. On the first component, evaluating real points at A will produce a non-trivial of $\mathrm{Br}(\mathbb{R})$, but on the second the resulting quaternion algebra will be $\mathbb{R}$-split. We conclude that $\mathbb{Q}$ points can only occur on one of the connected real components.

5 Insufficiency of Brauer-Manin Obstruction

Following [2] chapter 14 Say we have the following

- k is a number field with a real place v
- C is a nice curve over k such that $C(k)$ consists of a single element P

- $\Pi \subset C(k_v)$ is an open interval containing P
- $f : C \to \mathbb{P}^1$ is a map chosen so that
  - $f(P) = \infty$ and f is unramified at P
  - f is unramified over 0
- $a > 0$ in k (we use v to define being positive) is such that f is unramified over $t = a$ and $a \in f(\Pi)$
- $Q(x_0, x_1, x_2)$ is a quadratic homogenous form over k that represents 0 at all places of k except v and one other place w. We may assume Q is positive definite at v.
- $n \in k$ with $n > 0$ and $-nQ(1, 0, 0) \in k_w^2$

We now construct a quadric bundle Y over $\mathbb{P}^1$. Over $\mathbb{P}^1\backslash\infty$ we take the bundle defined by

$$Q(x_0, x_1, x_2) + nt(t - a)x_3^2 = 0$$

Here t is the coordinate on $\mathbb{A}^1 = \mathbb{P}^1\backslash\infty$

Over $\mathbb{P}^1\backslash 0$ we take the bundle defined by

$$Q(x_0, x_1, x_2) + n(1 - aT)X_3^2 = 0$$

Over $\mathbb{P}^1\backslash\{0, \infty\}$ we glue the two bundles together by taking $t = T^{-1}$ and $X_3 = tx_3$. This produces a smooth quadric bundle Y with degenerate fibers only over $t = 0$ and $t = a$.

Now take $X = Y \times_{\mathbb{P}^1} C$. $p : X \to C$ is a smooth "quadric bundle".

We first observe that $X(k) = \emptyset$. SInce $C(k) = \{P\}$, any k-points of X must lie in X_P, the fiber over P. But this is the conic

$$Q(x_0, x_1, x_2) + nX_3^2 = 0$$

And this quadric has no k_v points since Q is k_v positive definite.

By assumption, X_P has k_η points for all η not equal to v or w, since Q represents 0 at those places. Since $-nQ(1, 0, 0)$ is a square in k_w, X_P has a k_w point.

We can find a k_v point of X by taking Q to be a point of C lying above a. X_Q is a quadric cone, and its singular point is a k_v point. We can take all of these points together to produce a point of $x \in X(\mathbb{A})$.

The idea is then to show that π is always 0 when paired with an element of $\mathrm{Br}(X)$. From [] we have

Proposition 4. *Let $f : X \to Y$ be a proper surjective morphism of smooth geometrically integral varieties over a field k of characteristic zero such that the generic fibre X_K is a smooth quadric of dimension at least 1. Suppose that either all the fibres over points of codimension 1 in Y are split, or dim$(X_K) \geq 3$. Then the map $f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is surjective*

This show that $\mathrm{Br}(C)$ surjects onto $\mathrm{Br}(X)$. So for any point x in $X(k_\eta)$ we can evaluate it with a Brauer class $A \in \mathrm{Br}(X)$ by taking a lift of A to $\mathrm{Br}(C)$ and pairing it with the image of x in $C(k_\eta)$.

$x_\eta \in (X_\eta)_P$ for $\eta \neq v$, and $P \in C(k)$ so for any $A \in \mathrm{Br}(X)$ $ev_A(x_\eta) = ev_A(P)$.

Furthermore, $p(x_v)$ lies in Π, so it lies in the same connected component of $C(k_v)$ as P does, so $ev_A(\pi_v) = ev_A(P)$ So then

$$\mathrm{inv}(ev_A(x)) = \sum_\eta \mathrm{inv}(A(x_\eta)) = \sum_\eta \mathrm{inv}(A(P)) = 0$$

This shows that x is contained in $X(\mathbb{A})^{Br}$.

References

[1] Bjorn Poonen *Rational Points on Varieties*

[2] Jean-Louis Colliot-Thélène Alexei N. Skorobogatov *The Brauer-Grothendieck Group*