**Definition 1.** A graph \( H \) is a minor of a graph \( G \) if it can be obtained from \( G \) by a sequence of edge contractions, edge deletions and vertex deletions.

**Example 1.** The Petersen graph contains \( K_{3,3} \) as a minor, shown with red/blue vertices and bold edges below:

![Diagram of Petersen graph containing \( K_{3,3} \)]

It also contains \( K_5 \) as a minor, as seen by contracting the bold green edges shown below:

![Diagram of Petersen graph containing \( K_5 \) by contracting green edges]

**Definition 2.** A graph \( H' \) is a subdivision of a graph \( H \) if it can be obtained from \( H \) by a sequence of edge subdivisions:

\[
\begin{array}{c}
\quad u \quad v \\
\quad \quad \rightarrow \\
\quad \quad u \quad w \quad v
\end{array}
\]
Example 2. The Petersen graph contains a subdivision of $K_{3,3}$, shown with red/blue vertices and bold edges below:

However, it does not contain any subdivision of $K_5$, simply because all the vertices have degree less than 4.

Theorem 1 (Wagner). A graph $G$ is non-planar if and only if it contains $K_5$ or $K_{3,3}$ as a minor.

Theorem 2 (Kuratowski). A graph $G$ is non-planar if and only if it contains some subdivision of $K_5$ or $K_{3,3}$.

In a sense, Kuratowski’s theorem is stronger. The existence of a subgraph $H$ which is a subdivision of $K_5$ or $K_{3,3}$ automatically implies that $G$ contains $K_5$ or $K_{3,3}$ as a minor (simply contract the subdivided edges in $H$).

The two main ingredients to prove Kuratowski’s theorem are:

Argument 1: If $G$ has no subdivisions of $K_5$ and $K_{3,3}$, and $e$ is an edge, then the contraction $G/e$ produces no subdivisions of $K_5$ and $K_{3,3}$.

Argument 2: (Inductive) Assume $G$ has $n$ vertices with no subdivision of $K_5$ and $K_{3,3}$, and $xy$ is an edge. Contract $x$ and $y$ into a single vertex $z$. Then contracted graph $G'$ with $n - 1$ vertices can be drawn in the plane. If we know that $z$ lies within a cycle in this, then we can replace $z$ by $x$ and $y$ without creating any crossing.

For full details, see Lemma 6.2.10 and Theorem 6.2.11 in West’s book.