I. Recall that a burst of length $b$ is a string $x \in \mathbb{Z}_p^n$ of the form:
\[
x = 0 \ldots 0 x_{i+1} \ldots x_{i+b} 0 \ldots 0_{n-i-b},
\]
where $x_{i+1}, x_{i+b} \neq 0$. Let $C \subseteq \mathbb{Z}_p^n$ be a $b$-burst error correcting, meaning that for any two codewords $c \neq d \in C$ and any two bursts $x, y \in \mathbb{Z}_p^n$ of length $\leq b$, we have:
\[
c + x \neq d + y.
\]

a) [5pts] Prove that all bursts of length $\leq b$ lie in different cosets of $C$.
b) [5pts] Count number of all bursts of length $\leq b$ in $\mathbb{Z}_p^n$ (in terms of $n$ and $p$).
c) [5pts] Prove that $k \leq n - b + 1 - \log_p \left( (n - b + 1)(p - 1) + 1 \right)$, where $k = \dim(C)$.

II. For each of the following linear codes $C$, find a basis for $C^\perp$:

a) [5pts] $C = \langle 110100, 001101 \rangle \subset \mathbb{Z}_2^6$.
b) [5pts] $C = \{ x \in \mathbb{Z}_p^{2k} : x_1 = x_{2k}, x_2 = x_{2k-1}, \ldots, x_k = x_{k+1} \}$.
c) [5pts] $C = \langle e_1 + e_{b+1}, e_2 + e_{b+2}, \ldots, e_{n-b} + e_n \rangle \subset \mathbb{Z}_p^n$, where $1 \leq b < n$.

III. a) [5pts] Let $C, D \subseteq \mathbb{Z}_p^n$ be linear codes with $C \subseteq D$, prove that $D^\perp \subseteq C^\perp$.
b) [5pts] Prove or disprove the following claim: $d(C)$ is odd if and only if $d(C^\perp)$ is odd.

IV. a) [5pts] Let $C \subset \mathbb{Z}_p^n$ be a self-dual code, prove that $n$ must be even.
b) [5pts] Construct an explicit self-dual code $C \subset \mathbb{Z}_2^n$ for each even $n$. 