Due on Tuesday December 11th. Work on the following problems. Each problem is worth 10 points. Start each solution on a new page.

I. Let \( A_1, \ldots, A_n \subseteq S \) be finite sets. Define:
\[
\Sigma_k := \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap \cdots \cap A_{i_k}|
\]
Show that:
\[
|A_1 \cup \cdots \cup A_n| \leq k \sum_{i=1}^k (-1)^{i-1} \Sigma_i \quad \text{if } k \text{ odd},
\]
\[
|A_1 \cup \cdots \cup A_n| \geq k \sum_{i=1}^k (-1)^{i-1} \Sigma_i \quad \text{if } k \text{ even}.
\]

II. (Problem 10G textbook) Count the number of permutations \( x_1, \ldots, x_{2n} \) of \( 1, \ldots, 2n \) such that \( x_i + x_{i+1} \neq 2n + 1 \) for \( i = 1, \ldots, 2n - 1 \).

III. (Problem 10C textbook) Determine \( \sum_{y \leq z} \mu(y, z) \).

IV. Recall that for a poset \( P = (X, \preceq) \), we defined the Möbius function as:
\[
\mu(x, x) = 1, \quad \sum_{y \preceq z \preceq x} \mu(y, z) = 0 \quad \text{if } y < x.
\]
Prove that we also have
\[
\sum_{y \preceq z \preceq x} \mu(z, x) = 0 \quad \text{if } y < x.
\]

V. Given two posets \( P = (X, \preceq) \) and \( Q = (Y, \preceq) \), their product \( P \times Q \) is the poset on pairs \((x, y) \in X \times Y \) with:
\[
(x_1, y_1) \preceq (x_2, y_2) \iff x_1 \preceq x_2 \text{ and } y_1 \preceq y_2.
\]
a) Show that the Möbius functions on \( P, Q \) and \( P \times Q \) satisfy the relation:
\[
\mu_{P \times Q}((x_1, y_1), (x_2, y_2)) = \mu_P(x_1, x_2) \mu_Q(y_1, y_2)
\]
b) Let \( P = C_1 \times \cdots \times C_m \), where each \( C_i = x_{i1} < \cdots < x_{im} \) is a chain of \( n \) elements. Compute the Möbius function on \( P \).

VI. Let \( G = (V, E) \) be a bipartite graph. Show that \( M = (E, \mathcal{I}) \), where
\[
\mathcal{I} = \{ S \subseteq E : E \text{ is a matching} \},
\]
is a matroid (called the matching matroid).

VII. Let \( M = (E, \mathcal{I}) \) be a matroid:

a) Let \( E' = \{ e \in E : \{ e \} \in \mathcal{I} \} \). We say that \( e_1, e_2 \in E' \) are parallel, denoted \( e_1 \parallel e_2 \), if \( \{ e_1, e_2 \} \notin \mathcal{I} \). Show that \( \parallel \) is an equivalence relation, i.e., \( e_1 \parallel e_2 \) and \( e_2 \parallel e_3 \) implies \( e_1 \parallel e_3 \).
b) Let $A, B \subseteq E$. If $r(A) = r(A \cup b)$ for every $b \in B$, show that $r(A) = r(A \cup B)$.

**VIII.** Prove that if $B_1$ and $B_2$ are bases of a matroid $M$ and $e \in B_1 \setminus B_2$, then there exists $f \in B_2 \setminus B_1$ such that both $(B_1 \setminus e) \cup f$ and $(B_2 \setminus f) \cup e$ are bases of $M$.

**IX.** Let $M = (E, I)$ be a matroid and $X \subseteq E$. The *closure* of $X$ is

$$\text{cl}(X) := \{x \in E : r(X \cup x) = r(X)\}.$$ 

A subset $X \subseteq E$ is called a *flat* if $\text{cl}(X) = X$. Prove that if $X, Y \subseteq E$ are flats, then so is $X \cap Y$.

**X.** If $M = (E, I)$ is a matroid of rank $n$, then a *hyperplane* is any flat of rank $n - 1$. Let $Y \subset X$ be flats in $M$ such that $r(X) = r(Y) + 1$. Show that there is a hyperplane $H$ such that $Y = H \cap X$. 