Due on Thursday November 8th. Work on the following problems. Each problem is worth 10 points. Start each solution on a new page.

I. Recall the Turán graph $T(n,k)$: its $n$ vertices are partitioned into $k - 1$ classes such that every two classes differ by at most one in size; its edges connect all possible pairs of vertices from two different classes. Denote by $|T(n,k)|$ its number of edges.
    Let $G = (V,E)$ be a simple graph on $n$ vertices avoiding $K_k$ such that $|E|$ is maximum, i.e., $|E| = |T(n,k)|$. Show that $G$ must be the Turán graph $T(n,k)$.

II. (Problem 4C textbook) Prove that if a simple graph on $n$ vertices has $e$ edges, then it has at least $\frac{e^3}{3n}(4e - n^2)$ triangles.

III. Prove that for any $t \geq 2$, if a graph $G$ on $n$ vertices contains no $K_{2,t}$ subgraphs then it has at most $(\sqrt{t-1}n^{3/2} + n)/2$ edges.

IV. Prove that if a graph $G$ on $n$ vertices contains no $K_{3,3}$ subgraphs then it has at most $cn^{2/3}$ edges, where $c$ is some constant.

V. (Problem 3E textbook) A tournament on $n$ vertices is an orientation of $K_n$ (refer to HW1). A transitive tournament is a tournament on which the vertices can be renumbered in such a way that $v_i \rightarrow v_j$ is an edge if and only if $i < j$.
    a) Show that if $k \leq \log_2 n$, every tournament on $n$ vertices contains a transitive sub-tournament on $k$ vertices.
    b) Show that if $k > 1 + 2\log_2 n$, there exists a tournament on $n$ vertices with no transitive subtournaments on $k$ vertices.

VI. (Problem 3F textbook) Prove that for all $r \in \mathbb{N}$, there is a minimal number $N(r)$ with the following property. If $n \geq N(r)$ and the integers $\{1, \ldots, N(r)\}$ are colored with $r$ colors, then there are three integers $x, y, z$ (not necessarily distinct) with the same color and $x + y = z$. Determine $N(2)$. Show that $N(3) > 13$.

VII. (Problem 3G textbook) Let $m \in \mathbb{N}$ be given. Show that if $n$ is large enough, every $n \times n$ $(0,1)$-matrix has a principal submatrix of size $m$ in which all below-diagonal elements are the same, and all above-diagonal elements are the same.

VIII. (Problem 3H textbook) Show that if the edges of $K_{17}$ are colored with three colors, there must be a monochromatic triangle.

IX. (Problem 3J textbook) The edges of $K_n$ are colored red/blue in such a way that every red edge is in at most one red triangle. Show that there is a subgraph $K_k$ with $k \geq \lceil \sqrt{2n} \rceil$ that contains no red triangles.