I. Find a binary Huffman encoding for:
   a) \( \mathbb{P} = \{ \frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \ldots, \frac{8}{36} \} \).
   b) \( \mathbb{P} = \{ \frac{1}{13}, \frac{1}{13}, \ldots, \frac{1}{13} \} \).

II. Let \( C \) be a code. Denote by \( \ell(C) \) the sequence of codeword lengths in \( C \) arranged in increasing order. For example, if \( C = \{ 01, 111, 10, 101, 0011 \} \) then \( \ell(C) = 2, 2, 3, 3, 4 \).
   a) Give an example of a source \( (S, \mathbb{P}) \) with two Huffman encodings \( (C_1, f_1), (C_2, f_2) \) such that \( \ell(C_1) \neq \ell(C_2) \).
   b) Which step in Huffman encoding algorithm can lead to \( \ell(C_1) \neq \ell(C_2) \)? Why?

III. Let \( (C, f) \) be a binary Huffman encoding for a source \( (S, \mathbb{P}) \). Suppose that the codewords in \( C \) have length \( \ell_1, \ldots, \ell_q \) with \( L = \max \{ \ell_i \} \). Prove that:
   a) \( \sum_{i=1}^{q} 1/2^{\ell_i} = 1 \).
   b) \( C \) contains two codewords of length \( L \), which only differ in their last bit.

IV. Fix a source alphabet \( S = \{ s_1, \ldots, s_q \} \). Let \( \mathbb{P} = \{ p_1, \ldots, p_q \} \) be a probability distribution on \( S \). Denote by \( \text{MinACL}(\mathbb{P}) \) the minimum average codeword length over all instantaneous binary encoding schemes for \( (S, \mathbb{P}) \). We already know this is achieved with Huffman encodings.
   a) Assume \( q = 2^\ell \) for some \( \ell \geq 1 \) and \( p_i + p_j \geq p_k \) for every \( 1 \leq i, j, k \leq q \). Prove that \( \text{MinACL}(\mathbb{P}) = \ell \).
   b) Now consider a general \( q \) (not necessarily a power of 2). What is \( \text{MinACL}(\mathbb{P}) \) if \( \mathbb{P} \) is the uniform distribution?

**Bonus:** Describe Huffman encoding for an arbitrary base \( r \geq 2 \). Prove that it is optimal. If the source has size \( q \), how many symbols do we pick in the first step?