Due on Thursday Sep 20. Work on the following problems. Each problem is worth 10 points. Start each solution on a new page.

I. Problem 2E (textbook).

II. A tree with no inversions on \([n]\) is one such that for every \(i \in [n]\), the path \(n \rightarrow i\) contains only decreasing labels \(n > a_1 > a_2 > \cdots > i\). Prove that the number of trees with no inversions on \([n]\) is \((n-1)!\).

III. a) Consider a labelled tree \(T\) on \([n]\). In the last step Prüfer bijection, only two vertices of \(T\) remain. One of them is always \(n\). How to quickly tell the other one just by looking at \(T\)? Prove your claim. Hint: look at \(n-1\).

b) Generalize part a): If we look at the neighbors \(i_1, \ldots, i_k\) of \(n\) in \(T\). How do we know in which order they disappear without performing the Prüfer bijection? Prove your claim.

IV. The complete bipartite graph \(K_{m,n}\) is the graph on vertices \(\{1, \ldots, m\}\) and \(\{1', 2', \ldots, n'\}\) with all possible edges \((i, j')\) for \(1 \leq i \leq m\) and \(1 \leq j \leq n\). Prove that the number of spanning trees of \(K_{m,n}\) is \(m^{n-1}n^{m-1}\). Hint: Develop a version of Prüfer code for \(K_{m,n}\).

V. a) Compute the expected number of leaves in a random labelled tree on \([n]\).

b) Compute the expected degree of \(n\) in a random labelled tree on \([n]\).

Hint: Linearity of expectation (see https://brilliant.org/wiki/linearity-of-expectation/).

VI. Problem 2D (textbook). Hint: contradiction, just like the proof of Kruskal.

VII. Let \(A = (a_{ij})\) be the adjacency matrix of a graph \(G\), i.e., \(a_{ii} = 0\) for all \(i\) and \(a_{ij} = a_{ji} = 1\) if \((i, j)\) is an edge. Prove that the \(ij\)-th entry in \(A^m\) counts the number of walks of length \(m\) from vertex \(i\) to vertex \(j\). Such a walk has the form \(i = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_m = j\), where each pair \((v_i, v_{i+1})\) is adjacent.

VIII. Let \(G\) be an undirected graph with an Eulerian circuit. Prove that every edge of \(G\) lies in an odd number of cycles.

IX. The complete digraph \(\vec{K}_n\) has all possible edges \(\vec{ij}\) for \(1 \leq i, j \leq n, i \neq j\). The complete bipartite digraph \(\vec{K}_{m,n}\) has all possible pairs of edges \(\vec{ij}'\) and \(\vec{j'i}\) for \(1 \leq i \leq m, 1 \leq j \leq n\). How many Eulerian circuits are there in \(\vec{K}_n\) and \(\vec{K}_{m,n}\)? Prove your claims.

X. A tournament on \([n]\) is an arbitrary orientation of \(K_n\), i.e., for every pair \((i, j)\), we either pick \(\vec{ij}\) or \(\vec{ji}\). Prove that every tournament has a directed path going through all the vertices. Hint: induction.