Sphere Method-7-6 Using No Matrix Inversions for Linear Programs (LPs)

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Abstract

Existing software implementations for solving Linear Programming (LP) models are all based on full matrix inversion operations involving every constraint in the model in every step. This linear algebra component in these systems makes it difficult to solve dense models even with moderate size, and it is also the source of accumulating roundoff errors affecting the accuracy of the output.

We present a new version of the Sphere method, SM-7-6, for LP not using any pivot steps.

Key words: Linear Programming (LP), solving LPs by descent feasible methods without using matrix inversions.

1 Sphere Method, SM-7-6, for LP

In 2006, Sphere methods for LP, IPMs based on the properties of spheres (instead of ellipsoids like in other IPMs) were introduced in Murty [2006a, b]. The initial version of the sphere method also needed pivot steps for matrix inversions, but these pivot steps only involve a subset
of constraints in the original LP. After some other versions, in this paper we describe SM-7-6, not involving any pivot steps.

SMs consider LPs in the form:

\[
\begin{align*}
\min & \quad z = cx \\
\text{subject to} & \quad Ax \geq b
\end{align*}
\] (1)

where \( A \) is an \( m \times n \) data matrix; with a known interior feasible solution \( x \) (i.e., satisfying \( Ax > b \)). LPs in any other form can be directly transformed into this form, see [Murty 2009a, b], Murty, Oskoorouchi [2010]. Here is some basic notation that we will use.

- **Notation for rows and columns of \( A \):** \( A_i, A_j \) denote the \( i \)th row, and \( j \)th column of \( A \). The index \( i \) has range \( 1 \leq i \leq m \), and \( j \) ranges in \( 1 \leq j \leq n \).

- **Feasible region and its interior:** \( K \) denotes the set of feasible solutions of (1), and \( K^0 = \{ x : Ax > b \} \) is its interior.

- **Facetal hyperplanes, and their half-spaces containing \( K \):** \( FH_i = \{ x : A_i x = b_i \} \), the \( i \)-th facetal hyperplane of \( K \) for \( i = 1 \) to \( m \). Also, \( FH^+_i = \{ x : A_i x \geq b_i \} \) is the half-space of \( FH_i \) containing \( K \).

- **IFS:** Interior feasible solution, a point \( x \in K^0 \)

- **\( \delta \)(x) :** Defined for \( x \in K \), it is the radius of the largest ball inside \( K \) with \( x \) as center. From Murty [2006a, b], we know that \( \delta(x) = \min \{ \frac{A_i x - b_i}{\|A_i\|} : i = 1, \ldots, m \} \). For any point \( x \) on the boundary of \( K \), i.e., satisfying at least one of the constraints in (1) as an equation, \( \delta(x) = 0 \) by this definition.

- **Largest ball inscribed in \( K \) with a given IFS \( x \) as center:** \( B(x) = \{ y : \|y - x\| \leq \delta(x) \} \) is that largest inscribed ball in \( K \) with \( x \) as its center.

- **Touching constraint index set at a given IFS:** \( T(x) \) defined for \( x \in K^0 \), is the set of all indices \( i \) satisfying: \( \frac{A_i x - b_i}{\|A_i\|} = \min \{ \frac{A_p x - b_p}{\|A_p\|} : p = 1 \text{ to } m \} = \delta(x) \). The facetal
hyperplane $FH_i = \{ x : A_i x = b_i \}$ is a tangent plane to $B(x)$ for each $i \in T(x)$, that’s why $T(x)$ is called the **index set of touching constraints** in (1) defining $K$, at $x$.

- **Touching point** $x^i$ : Defined for $x \in K^0$ and $i \in T(x)$, it is the nearest point on $FH_i$ to $x$, it is the orthogonal projection $x - A_i^T(A_i x - b_i)/||A_i||^2$ of $x$ on $FH_i$. It is the point where the ball $B(x)$ touches $FH_i$ for $i \in T(x)$.

- **$H(\hat{x})$** : Defined for any feasible solution $\hat{x} \in K$, $H(\hat{x}) = \{ x : cx = c\hat{x} \}$ is the objective plane through $\hat{x}$

- **$\tilde{x}$** : Defined for any IFS $\hat{x} \in K^0$, it is $\hat{x} - \delta(\hat{x})c^T/||c||$ = the bottom point of $B(\hat{x})$ in the direction $-c^T$, the point where the objective plane touches $B(\hat{x})$ when it is moved down from its present position $H(\hat{x})$, in the direction $-c^T$ until it becomes a tangent plane to $B(\hat{x})$

- **$\tilde{\hat{x}}$** : Defined for any IFS $\hat{x} \in K^0$ and $i \in T(\hat{x})$, it is $\hat{x}^i - c^T[(c\hat{x}^i - c\tilde{x})/cc^T]$ = the orthogonal projection of $\hat{x}^i$ on $H(\tilde{x})$. 

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Figure 1: $\hat{x}$ is an IFS of $K$, $B(\hat{x})$ is the largest sphere with center $\hat{x}$ as center inside $K$. $\tilde{x}$ is the point in $B(\hat{x})$ with the smallest value for $cx$, and $H(\tilde{x})$ is the objective plane through $\tilde{x}$, it is the tangent plane to $B(\hat{x})$ at $\tilde{x}$. Facets 1, 2 of $K$ are tangent planes to $B(\hat{x})$ with touching points $\hat{x}^1, \hat{x}^2$ respectively, so $T(\hat{x}) = \{1, 2\}$. $\vec{\tilde{x}}^1$ is the orthogonal projection of $\hat{x}^1$ on $H(\tilde{x})$. Thanks to Madhusri Katta, Vijaya Katta for Figures 1, 3.

SM-7-6 is based on feasible descent steps (starting with a feasible solution, maintaining feasibility throughout, with objective value improving monotonically), but not using any pivot steps at all. The 1st iteration begins with the given IFS $\hat{x}$, all subsequent iterations begin with the best solution (by objective value) obtained in the descent steps in the previous iteration.

We discuss a simplified version of the algorithm discussed in SM-7-3. Also, to keep the discussion simple, we first restrict the discussion to the case in which $K$ is a polytope, i.e., it is
bounded.

2 A General Iteration in the Sphere Method, SM-7-6, for the LP (1)

In every iteration of SM-7-6, we face a problem of finding the interval of values of a real parameter \( \nu \) say, satisfying a given system of linear inequalities in the parameter. Now we give the procedure, we will call it Subroutine 1 for computing this interval.

Subroutine 1: Let the system of inequalities in \( \nu \) be

\[
a_t + g_t \nu \geq 0, \quad t = 1, \ldots, \ell
\]

In systems like this that we encounter in SM-7-6; for any \( t \) if \( g_t = 0, \) \( a_t \) will be \( \geq 0, \) and hence that constraint is a redundant constraint in the system. Let

\[
\nu^1 = \max \left\{ \left( -\frac{a_t}{g_t} \right) : \text{over all } t \text{ satisfying } g_t > 0 \right\}, \quad \nu^2 = \min \left\{ \left( -\frac{a_t}{g_t} \right) : \text{over all } t \text{ satisfying } g_t < 0 \right\}
\]

Here define the maximum [minimum] in the empty set to be \(-\infty[+\infty]\) respectively. If \( \nu^1 > \nu^2 \) system (2) has no solution. Otherwise the required interval for \( \nu \) feasible to this system is \( \nu^1 \leq \nu \leq \nu^2. \)

Also in every iteration of this algorithm, we encounter the problem of solving a 2-variable LP in the variables \( \alpha, \lambda \) of the following form:

\[
\begin{align*}
\text{Maximize } & \lambda \\
\text{Subject to } & c_p \alpha + d_p \lambda \geq g_p, \quad p = 1 \text{ to } m, \\
& \alpha \text{ real, and } \lambda \geq 0
\end{align*}
\]
starting with a given feasible solution \((\alpha_1, \lambda_1)\). Now we give the procedure, we will call it Subroutine 2, for solving this 2-variable LP. Here we discuss the version of Subroutine 2 in the case where the set of feasible solutions of \((3)\) is bounded. The version of Subroutine 2 when the set of feasible solutions of \((3)\) may be unbounded is discussed later in Section 3.

**Subroutine 2:** Each step in this algorithm to solve the 2-variable LP \((3)\) consists of three substeps. We will describe the general \(r\)-th step in this algorithm beginning with the feasible solution \((\alpha_r, \lambda_r)\).

**Substep 1:** Fix \(\lambda = \lambda_r\), find the interval of feasibility for \(\alpha\) with it. Let \((\alpha_r, \lambda_r)\) be its midpoint. If this interval of feasibility consists of the single point \((\alpha_r, \lambda_r)\), then this point \((\alpha_r, \lambda_r)\) is the optimum solution of \((3)\), terminate. Otherwise continue.

Now fix \(\alpha = \alpha_r\), find \(\lambda_r\) the maximum value of \(\lambda\) feasible to \((3)\) with it using Subroutine 1. With the feasible solution \((\alpha_r, \lambda_r)\) go to Substep 2.

**Substep 2:** Fix \(\lambda = \lambda_r\), and find the midpoint \((\alpha_r, \lambda_r)\) of the interval of feasibility to \((3)\) with it. If this point is the only point in this interval, then it is the optimum solution of \((3)\), terminate. Otherwise, continue.

Now fix \(\alpha = \alpha_r\), find \(\lambda_r\) the maximum value of \(\lambda\) feasible to \((3)\) with it, using Subroutine 1. With the feasible solution \((\alpha_r, \lambda_r)\) go to Substep 3.

**Substep 3:** The half-line joining the point \((\alpha_r, \lambda_r)\) to \((\alpha_r, \lambda_r)\) and continuing in the same direction has the general point \((\alpha_r + \gamma(\alpha_r - \alpha_r), \lambda_r + \gamma(\lambda_r - \lambda_r))\), where \(\gamma \geq 0\) is a nonnegative parameter. The last point on this half-line feasible to \((3)\) is this general point corresponding to \(\gamma = \gamma^*\), where this \(\gamma^*\) is the maximum value of \(\gamma\) for which this general point is feasible to \((3)\).

Among the pair of points \((\alpha_r, \lambda_r), (\alpha_r + \gamma^*(\alpha_r - \alpha_r), \lambda_r + \gamma^*(\lambda_r - \lambda_r))\) denote the one corresponding to the maximum value for the \(\lambda\)-coordinate in it by \((\alpha_{r+1}, \lambda_{r+1})\).

If the difference \(\lambda_{r+1} - \lambda_r\) is smaller than some small tolerance, terminate the algorithm with \((\alpha_{r+1}, \lambda_{r+1})\) as a near optimum solution of the 2-variable LP \((3)\). Otherwise, with \((\alpha_{r+1}, \lambda_{r+1})\) as the initial feasible solution go to the next step in the algorithm.

Terminate this process when the difference between the values of \(\lambda\) in consecutive points in the sequence becomes \(\leq \epsilon\), where \(\epsilon\) is a small positive tolerance value. If \((\alpha_s, \lambda_s)\) is the final
point in the sequence generated, choose the feasible point among \((\alpha_s \pm \epsilon, \lambda_s - \epsilon)\) as the near optimum solution of (3).

**Figure 2:** An illustration of the feasible region of (3). Here \(\alpha\) is plotted on the horizontal axis, and \(\lambda\) is plotted along the vertical axis. Starting with the feasible solution \((\alpha_r, \lambda_r)\), the output points in Substeps 2, 3 respectively in this Step of the algorithm are \(p = (\alpha_r, \lambda_r), q = (\alpha_r + \gamma^*(\alpha_r - \alpha_r), \lambda_r + \gamma^*(\lambda_r - \lambda_r))\). Among these two points, the one with a higher value for \(\lambda\), here \(q = (\alpha_r + \gamma^*(\alpha_r - \alpha_r), \lambda_r + \gamma^*(\lambda_r - \lambda_r))\) denoted by \((\alpha_{r+1}, \lambda_{r+1})\) is the beginning.
point for the next step in the algorithm. Thanks to Olof Minto for his help on Figures 2, 3.

Now we will describe the general iteration in SM-7-6 in the case when $K$ is bounded.

**General iteration beginning with the initial IFS $\hat{x}$**

Find $\delta(\hat{x}), T(\hat{x}), \bar{\hat{x}}$. If $\bar{\hat{x}}$ is a boundary point of $K$, i.e. satisfies $A_i \bar{\hat{x}} = b_i$ for some $i = 1$ to $m$, then $H(\bar{\hat{x}})$ must be the same as $\{ x : A_i x = b_i \}$, so $\bar{\hat{x}}$ is an optimum solution of the original LP, terminate the algorithm with this conclusion.

Otherwise $\bar{\hat{x}}$ is an IFS of $K$, continue.

For each $i \in T(\hat{x})$ find using Subroutine 1, the interval of values of the parameter $\alpha$ satisfying

$$A_p(\alpha \bar{\hat{x}} + (1 - \alpha)\hat{x}^i) \geq b_p \quad \text{for } p = 1 \text{ to } m.$$

This interval includes $0 \leq \alpha \leq 1$ since for these values of $\alpha$ the point $\alpha \bar{\hat{x}} + (1 - \alpha)\hat{x}^i$ is on the line segment joining $\hat{x}^i$ to $\bar{\hat{x}}$. Suppose this interval is $0 \leq \alpha \leq \alpha^2$. For each $i \in T(\hat{x})$, $\hat{x}^{i^2} = \alpha^2 \bar{\hat{x}} + (1 - \alpha^2)\hat{x}^i$ is the other boundary point of $K$ on the straight line joining $\hat{x}^i$ to $\bar{\hat{x}}$.

Let $r$ be the value of $i$ attaining the minimum in minimum $\{ c\hat{x}^{i^2} : i \in T(\hat{x}) \}$
Figure 3: $\hat{x}$ is an IFS of $K$, $B(\hat{x})$ is the largest sphere with center $\hat{x}$ as center inside $K$. $\bar{\hat{x}}$ is the point in $B(\hat{x})$ with the smallest value for $cx$, and $H(\bar{\hat{x}})$ is the objective plane through $\bar{\hat{x}}$, it is the tangent plane to $B(\hat{x})$ at $\bar{\hat{x}}$. Facets 1, 2 of $K$ are tangent planes to $B(\hat{x})$ with touching points $\hat{x}_1, \hat{x}_2$ respectively, so $T(\hat{x}) = \{1, 2\}$. The lines joining $\hat{x}_1, \hat{x}_2$ to $\bar{\hat{x}}$ intersect the boundary of $K$ again at $\hat{x}_{12}, \hat{x}_{22}$ respectively. $c\hat{x}_{12} = \min\{c\hat{x}_{12}, c\hat{x}_{22}\}$. So the index $r$ defined above
is 1 in this example.

The objective plane through the point \( \hat{x}r^2 \) is
\[
H(\hat{x}r^2) = \{ x : cx = c\hat{x}r^2 \}.
\]

The orthogonal projection from \( \hat{x}r \) on \( H(\hat{x}r^2) \) is \( y_1 = \hat{x}r - \alpha r c^T \) where \( \alpha_r \) is determined from the equation \( c(\hat{x}r - \alpha r c^T) = c\hat{x}r^2 \), and hence \( \alpha_r = (c\hat{x}r - c\hat{x}r^2) / cc^T \).

The half-line \( \{ x : x = \hat{x}r - \lambda c^T, \lambda \geq 0 \} \) intersects the objective plane \( H(\hat{x}r^2) \) at the point \( y_1 \).

Let \( L^{22} = \{ \beta \hat{x}r^2 + (1 - \beta)y_1 - \lambda c^T : \beta \) real, \( \lambda \geq 0 \} \), the 2-dimensional half-space determined by the line joining \( \hat{x}r^2 \) and \( y_1 \), and the direction \( -c^T \).

In \( L^{22} \cap K \), since \( cy_1 = c\hat{x}r^2 \), minimizing \( cx \) on it is the same as minimizing \( (c\hat{x}r^2 - \lambda cc^T) \), and hence this problem is equivalent to the following problem:

Maximize \( \lambda \)

Subject to \( A[\beta \hat{x}r^2 + (1 - \beta)y_1 - \lambda c^T] \geq b \)
\( \lambda \geq 0, \beta \) real

which can be solved by applying Subroutine 2. If \( (\hat{\beta}, \hat{\lambda}) \) is the solution of this 2-variable LP, then in terms of the original variables \( x^2 = \hat{\beta}\hat{x}r + (1 - \hat{\beta})\hat{x}r^2 - \hat{\lambda}c^T \) is the final solution obtained in this iteration. This may not be an IFS.

Let \( L \) be the straight lie through \( \tilde{x} \) parallel to the straight line joining \( \hat{x}r^2 \) and \( y_1 \); and suppose \( L^1 \) and \( L^2 \) are the end points of \( L \) in \( K \).

We now have two choices to determine a point \( \hat{x} \) on \( L \); the best among them has to be determined by computational tests to find out which one gives better results:

CHOICE 1: \( \hat{x} \) is the point on \( L \) intersected with \( K \) which has the maximum value of \( \delta(x) \)

CHOICE 2: \( \hat{x} \) is \( (L^1 + L^2)/2 \)

Take \( (x^2 + \hat{x})/2 \), the midpoint joining \( x^2 \) and \( \hat{x} \), this will be an IFS of \( K \); and \( H((x^2 + \hat{x})/2) \) is the objective plane through that point. Add the constraint \( cx \leq c((x^2 + \hat{x})/2) \) as the \( (m+1) \)th constraint to the original LP. This gives the problem
\[
\begin{align*}
\text{min} & \quad z = cx \\
\text{subject to} & \quad Ax \geq b \quad (4) \\
& \quad cx \leq c((x^2 + \hat{x})/2)
\end{align*}
\]

The objective plane \(H((x^2 + \hat{x})/2)\) divides \(\mathbb{R}^n\) into two half-spaces, of which \(H_+(x^2 + \hat{x})/2\) is the half-space in the direction \(-c^T\). \(K_+ = K \cap H_+(x^2 + \hat{x})/2\) is the set of feasible solutions of (4); and hence the set of optimum solutions of (1) and (4) are the same.

The point \((1/2)(x^2 + (x^2 + \hat{x})/2)\) is an IFS of \(K_+\). With that as the initial IFS, repeat the application of this algorithm to solve the LP (4). Continue this way. When you repeat this process, the problem corresponding to (4) is just (4) with a smaller RHS constant in the \((m+1)\)th constraint of it.

The repetition process is continued until the difference between objective valus of \(cx^2\) in consecutive iterations of the algorithm becomes smaller than a small \(\epsilon\). The final point corresponding to \(x^2\) in the final repetition is taken as an approximate optimum solution of the original problem (1).

3 Changes for the General Case When an Initial IFS in \(K\) Is Not Given, and When \(K\) is Not Known to be Bounded

We now consider the general LP model in nonnegative variables. If the model being solved includes \(\leq\) inequality constraints, \(D_iy \leq d_i\), replace each of them with the equivalent \(-D_iy \geq -d_i\).

Also, suppose the model being solved has \(r\) equality constraints

\[D_iy = d_i, \quad i = 1, \ldots, r\]

Then replace these \(r\) equations in the model by \(r\) inequalities

\[D_iy \geq d_i, \quad i = 1, \ldots, r\]
and then include $\bar{y} = \sum_{i=1}^r (D_i y - d_i)$ to the objective function being minimized, with a large positive cost coefficient. In the optimum solution of the resulting model if $\bar{y}$ has a positive value, it is an indication that the original model with the equality constraints is infeasible.

So, with these modifications we can transform the general LP model into the form (1).

1. So let us now consider the LP (1), but suppose an initial IFS for it is not available. Then we consider the modified problem:

Minimize $z = c_1 x_1 + ... + c_n x_n + c_{n+1} x_{n+1}$ \hspace{1cm} (5)
subject to $Ax + e x_{n+1} \geq b$
and $x_{n+1} \geq 0, \quad -x_{n+1} \geq b_{n+1},$

where $x_{n+1}$ is a nonnegative artificial variable, and $e$ is the column vector in $\mathbb{R}^m$ with all entries $= 1$. Let $\bar{x}_{n+1} = \delta_1 + \text{maximum}\{0, b_i : i = 1, ..., m\}$, and $b_{m+1} = -\bar{x}_{n+1} - \delta_2$, where $\delta_1, \delta_2$ are strictly positive numbers. $c_{n+1}$ is a large positive cost coefficient of $x_{n+1}$ in (5).

We will now consider (5) as the problem to solve, and continue to denote the new column vector of decision variables $x_1, ..., x_n, x_{n+1}$ by the same symbol $x$; its cost vector $(c_1, ..., c_n, c_{n+1})$ by the same symbol $c$. The column vector $\bar{x}$ with $x_1 = ... = x_n = 0$ and $x_{n+1} = \bar{x}_{n+1}$ is an IFS for (5). We can solve (5) beginning with this initial IFS by the algorithm discussed earlier, and since $c_{n+1}$ is a large positive cost coefficient, $x_{n+1}$ will have value 0 in its optimum solution, if the original LP (1) has an optimum solution.

2. Now consider the case in which $K$, the set of feasible solutions of the original LP, is not known to be bounded. In this case, since the feasibility set of the 2-variable LP of the form (3) encountered in iterations of the algorithm may be unbounded, some changes have to be made in Subroutine 2 used to solve it. Below, we discuss the modified Subroutine 2 used to solve it.

Subroutine 2 for the case when the set of feasible solutions of (3) is not known to be bounded

Let $(\alpha_1, \lambda^*_1)$ be the initial feasible solution of (3) available. Go to the new Sustep 0.

Substep 0: Fixing $\alpha = \alpha_1$ find the maximum value of $\lambda$ in the set of feasible solutions for (3). If it is $\infty$, then the maximum value of $\lambda$ in (3) is $+\infty$, and $\{(\alpha_1, \lambda) : \lambda \geq \lambda^*_1\}$ is a feasible
half-line for (3) along which \( \lambda \) diverges to \( \infty \), terminate.

Otherwise, let \( \lambda_1 \) be the maximum value of \( \lambda \) in (3) when \( \alpha \) is fixed at \( \alpha_1 \). With the feasible solution \((\alpha_1, \lambda_1)\) as the initial feasible solution go to Substep 1. Considering the general \( r \)-th step, let \((\alpha_r, \lambda_r)\) be the initial feasible solution.

**Substep 1:** When we fix \( \lambda = \lambda_r \) in (3), suppose the interval of feasibility for \( \alpha \) is \( \alpha_r \leq \alpha \leq \infty \). Now check how the maximum value of \( \lambda \) varies as a function of \( \alpha \) as it varies from \( \alpha_r \) to \( \infty \).

If after some value of \( \alpha \) in this interval, say \( \alpha = \bar{\alpha}_r \), \( \lambda \) remains constant at \( \bar{\lambda}_r \), then \((\bar{\alpha}_r, \bar{\lambda}_r)\) is an optimum solution of (3). On the other hand, if the maximum value of \( \lambda \) in (3) keeps going up as \( \alpha \) varies from \( \alpha_r \) to \( \infty \), then \( \lambda \) is unbounded in (3), and you can get a half-line along which it diverges to \( \infty \) from this.

On the other hand, with \( \lambda \) fixed at \( \lambda_r \) in (3), if the interval of feasibility for \( \alpha \) is bounded, let \((\alpha_{r1}, \lambda_r)\) be its midpoint. If this interval of feasibility consists of the single point \((\alpha_{r1}, \lambda_r)\), then this point \((\alpha_{r1}, \lambda_r)\) is the optimum solution of (3), terminate. Otherwise, fix \( \alpha = \alpha_{r1} \), and find \( \lambda_{r1} \), the maximum value of \( \lambda \) feasible to (3) with it, using Subroutine 1. With the feasible solution \((\alpha_{r1}, \lambda_{r1})\) go to Substep 2.

**Substep 2:** Fix \( \lambda = \lambda_{r1} \). With it, if the interval of feasibility for \( \alpha \) in (3) is \((\alpha_{r1}, \infty)\); check how the maximum value of \( \lambda \) in the feasibility interval of (3) varies as \( \alpha \) varies in this interval. If after some value of \( \alpha \), say \( \alpha_{r2}^2 \), in this interval, it remains constant at \( \lambda_{r1}^2 \), then \((\alpha_{r1}^2, \lambda_{r1}^2)\) is an optimum solution of (3).

On the other hand, if the maximum value of \( \lambda \) in (3) keeps going up as \( \alpha \) varies from \( \alpha_{r1} \) to \( \infty \), then \( \lambda \) is unbounded in (3), and you can get a half-line along which it diverges to \( \infty \) from this.

With \( \lambda \) fixed at \( \lambda_{r1} \) in (3) if the interval of feasibility for \( \alpha \) is bounded, let \((\alpha_{r2}, \lambda_{r1})\) be its midpoint. If this interval of feasibility consists of the single point \((\alpha_{r2}, \lambda_{r1})\), then this point \((\alpha_{r2}, \lambda_{r1})\) is the optimum solution of (3), terminate. Otherwise, fix \( \alpha = \alpha_{r2} \) in (3), and let \( \lambda_{r2} \) the maximum value of \( \lambda \) in it. With the feasible solution \((\alpha_{r2}, \lambda_{r2})\) go to Substep 3.

**Substep 3:** The half-line joining the point \((\alpha_{r1}, \lambda_r)\) to \((\alpha_{r2}, \lambda_{r1})\) and continuing in the same direction has the general point \((\alpha_{r1} + \beta(\alpha_{r2} - \alpha_{r1}), \lambda_r + \beta(\lambda_{r1} - \lambda_r))\), where \( \beta \geq 0 \) is a nonnegative parameter. The last point on this half-line feasible to (3) is this general point.
corresponding to $\beta = \beta^*$, where this $\beta^*$ is the maximum value of $\beta$ for which this general point is feasible to (3).

If $\beta^* = \infty$, then the half-line $\{(\alpha r_1 + \beta(\alpha r_2 - \alpha r_1), \beta(\lambda r_1 - \lambda r_)) : \beta \geq 0\}$ is a feasible half-line along which $\lambda$ diverges to $\infty$.

If $\beta^*$ is finite, among the pair of points $(\alpha r_2, \lambda r_2), (\alpha r_1 + \beta^*(\alpha r_2 - \alpha r_1), \lambda r + \beta^*(\lambda r_1 - \lambda r))$ denote the one corresponding to the maximum value for the $\lambda$-coordinate in it by $(\alpha r_1, \lambda r_1)$.

If the difference $\lambda r_1 - \lambda r$ is smaller than some small tolerance, terminate the algorithm with $(\alpha r_1, \lambda r_1)$ as a near optimum solution of the 2-variable LP (3). Otherwise, with $(\alpha r_1, \lambda r_1)$ as the initial feasible solution go to the next step in the algorithm.

Terminate this process when the difference between the values of $\lambda$ in consecutive points in the sequence becomes $\leq \epsilon$, where $\epsilon$ is a small positive tolerance value. If $(\alpha s, \lambda s)$ is the final point in the sequence generated, choose the feasible point among $(\alpha s \pm \epsilon, \lambda s - \epsilon)$ as the near optimum solution of (3).

**Changes in the General Iteration Beginning with the Initial IFS $\hat{x}$**

The work here is carried out as described earlier.

For each $i \in T(\hat{x})$ find using Subroutine 1, the interval of values of the parameter $\alpha$ satisfying

$$A_p(\alpha \bar{x} + (1 - \alpha)\hat{x}^i) \geq b_p \quad \text{for } p = 1 \text{ to } m.$$  

This interval includes $0 \leq \alpha \leq 1$ since for these values of $\alpha$ the point $\alpha \bar{x} + (1 - \alpha)\hat{x}^i$ is on the line segment joining $\hat{x}^i$ to $\bar{x}$. Suppose this interval is $0 \leq \alpha \leq \alpha i^2$. If for some $i \in T(\hat{x})$, this $\alpha i^2 = \infty$, then for that $i$, $\{\hat{x}^i + \alpha(\bar{x} - \hat{x}^i), \alpha \geq 0\}$ is a half-line in $K$ along which the objective function $cx$ diverges to $-\infty$, we terminate the algorithm with this conclusion.

Otherwise, for each $i \in T(\hat{x})$, $\hat{x}^i = \alpha i^2 \bar{x} + (1 - \alpha i^2)\hat{x}^i$ is the other boundary point of $K$ on the straight line joining $\hat{x}^i$ to $\bar{x}$.

Let $r$ be the value of $i$ attaining the minimum in minimum $\{c\hat{x}^i : i \in T(\hat{x})\}$

Now continue as before.

**Summary**
In each step the Classical Simplex Method for linear programming minimizes the objective function across a 1-dimensional boundary face of $K$, and for this work it needs to carry out a pivot step for changing the basis matrix by one column.

In this algorithm, in each step we minimize the objective function on a 2-dimensional polyhedron in the interior of $K$, that is the intersection of $K$ with a 2-dimensional hyperplane; and all this work is carried out using no pivot steps at all. I believe that this new method will produce results which are much better than those obtained under the simplex method.

**Computational results**
4 References


3. K. G. Murty, “Linear Equations, Inequalities, Linear Programs (LPs), and a New efficient Algorithm”, 1-36 in *Tutorials in OR*, INFORMS, 2006b.


7. K. G. Murty, Sphere Methods Using no Matrix Inversions for LPs and Extension to NLP, 0-1 ILP, on my webpage.