Sphere Method-7-6 Using No Matrix Inversions for Linear Programs(LPs)

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Abstract

Existing software implementations for solving Linear Programming (LP) models are all based on full matrix inversion operations involving every constraint in the model in every step. This **linear algebra component** in these systems makes it difficult to solve dense models even with moderate size, and it is also the source of accumulating roundoff errors affecting the accuracy of the output.

We present a new version of the Sphere method, SM- 7-6, for LP not using any pivot steps.

Key words: Linear Programming (LP), solving LPs by descent feasible methods without using matrix inversions.

1 Sphere Method, SM-7-6, for LP

In 2006, Sphere methods for LP, IPMs based on the properties of spheres (instead of ellipsoids like in other IPMs) were introduced in Murty [2006a, b]. The initial version of the sphere method also needed pivot steps for matrix inversions, but these pivot steps only involve a subset of constraints in the original LP. After some other versions, in this paper we describe SM-7-6, not involving any pivot steps.

SMs consider LPs in the form:

$$\begin{aligned} \min \quad & z = cx \\ \text{subject to} \quad & Ax \geq b \end{aligned} \tag{1}$$

where A is an $m \times n$ data matrix; with a known interior feasible solution x (i.e., satisfying $Ax > b$). LPs in any other form can be directly transformed into this form, see [Murty 2009a, b], Murty, Oskoorouchi [2010]. Here is some basic notation that we will use.

- **Notation for rows and columns of A:** $A_{i.}, A_{.j}$ denote the i^{th} row, and j^{th} column of A. The index i has range $1 \leq i \leq m$, and j ranges in $1 \leq j \leq n$.

- **Feasible region and its interior:** K denotes the set of feasible solutions of (1), and $K^0 = \{x : Ax > b\}$ is its interior.

- **Facetal hyperplanes, and their half-spaces containing K:** $FH_i = \{x : A_{i.}x = b_i\}$, the i-th facetal hyperplance of K for $i = 1$ to m. Also, $FH_i^+ = \{x : A_{i.}x \geq b_i\}$ is the half-space of FH_i containing K.

- **IFS:** Interior feasible solution, a point $x \in K^0$

- $\delta(x)$: Defined for $x \in K$, it is the radius of the largest ball inside K with x as center. From Murty [2006a, b], we know that $\delta(x) = \text{minimum}\{\frac{A_{i.}x - b_i}{||A_{i.}||} : i = 1, ..., m\}$. For any point x on the boundary of K , i.e., satisfying at least one of the constraints in (1) as an equation, $\delta(x) = 0$ by this definition.

- **Largest ball inscribed in K with a given IFS x as center:** $B(x) = \{y : ||y - x|| \leq \delta(x)\}$ is that largest inscribed ball in K with x as its center.

- **Touching constraint index set at a given IFS:** $T(x)$ defined for $x \in K^0$, is the set of all indices i satisfying: $\frac{A_{i.}x - b_i}{||A_{i.}||} = \text{Minimum}\{\frac{A_{p.}x - b_p}{||A_{p.}||} : p = 1 \text{ to } m\} = \delta(x)$. The facetal hyperplane $FH_i = \{x : A_{i.}x = b_i\}$ is a tangent plane to $B(x)$ for each $i \in T(x)$, that's why $T(x)$ is called the **index set of touching constraints in (1) defining** K, at x.

- **Touching point** x^i **:** Defined for $x \in K^0$ and $i \in T(x)$, it is the nearest point on FH_i to x, it is the orthogonal projection $x - A_{i.}^T(A_{i.}x - b_i)/||A_{i.}||^2$ of x on FH_i. It is the point where the ball $B(x)$ touches FH_i for $i \in T(x)$.

- $H(\hat{x})$: Defined for any feasible solution $\hat{x} \in K$, $H(\hat{x}) = \{x : cx = c\hat{x}\}$ is the objective plane through $\hat{x}$

- $\bar{\hat{x}}$: Defined for any IFS $\hat{x} \in K^0$, it is $= \hat{x} - \delta(\hat{x})c^T/||c|| =$ the bottom point of $B(\hat{x})$ in the direction $-c^T$, the point where the objective plane touches $B(\hat{x})$ when it is moved down from its present position $H(\hat{x})$, in the direction $-c^T$ until it becomes a tangent plane to $B(\hat{x})$

- $\bar{\hat{x}}^i$: Defined for any IFS $\hat{x} \in K^0$ and $i \in T(\hat{x})$, it is $= \hat{x}^i - c^T[(c\hat{x}^i - c\bar{\hat{x}})/cc^T] =$ the orthogonal projection of $\hat{x}^i$ on $H(\bar{\hat{x}})$.

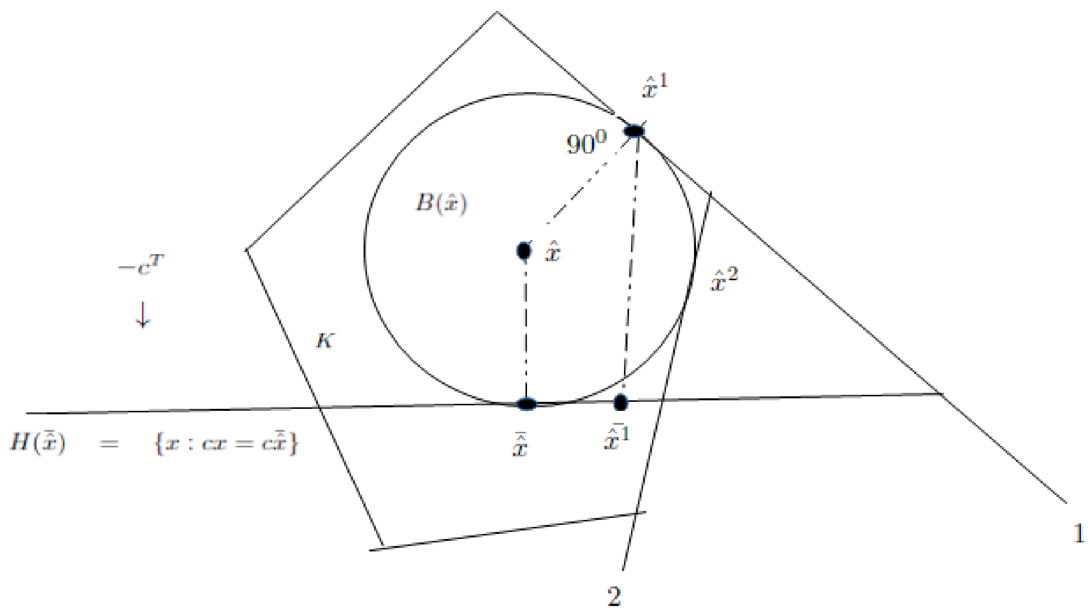

Figure 1: $\hat{x}$ is an IFS of K, $B(\hat{x})$ is the largest sphere with center $\hat{x}$ as center inside K. $\bar{\hat{x}}$ is the point in $B(\hat{x})$ with the smallest value for cx, and $H(\bar{\hat{x}})$ is the objective plane through $\bar{\hat{x}}$, it is the tangent plane to $B(\hat{x})$ at $\bar{\hat{x}}$. Facets 1, 2 of K are tangent planes to $B(\hat{x})$ with touching points $\hat{x}^1, \hat{x}^2$ respectively, so $T(\hat{x}) = \{1, 2\}$. $\bar{\hat{x}}^1$ is the orthogonal projection of $\hat{x}^1$ on $H(\bar{\hat{x}})$. Thanks to Madhusri Katta, Vijaya Katta for Figures 1, 3.

SM-7-6 is based on feasible descent steps (starting with a feasible solution, maintaining feasibility throughout, with objective value improving monotonically), but not using any pivot steps at all. The 1st iteration begins with the given IFS $\hat{x}$, all subsequent iterations begin with the best solution (by objective value) obtained in the descent steps in the previous iteration.

We discuss a simplified version of the algorithm discussed in SM-7-3. Also, to keep the discussion simple, we first restrict the discussion to the case in which K is a polytope, i.e., it is bounded.

2 A General Iteration in the Sphere Method, SM-7-6, for the LP (1)

In every iteration of SM- 7-6, we face a problem of finding the interval of values of a real parameter ν say, satisfying a given system of linear inequalities in the parameter. Now we give the procedure, we will call it **Subroutine 1** for computing this interval.

Subroutine 1: Let the system of inequalities in ν be

$$a_t + g_t \nu \geq 0, \qquad t = 1, ..., \ell \tag{2}$$

In systems like this that we encounter in SM- 7-6; for any t if $g_t = 0$, a_t will be ≥ 0, and hence that constraint is a redundant constraint in the system. Let

$$\begin{aligned} \nu^1 &= \text{maximum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t > 0\} \\ \nu^2 &= \text{minimum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t < 0\} \end{aligned}$$

Here define the maximum [minimum] in the empty set to be $-\infty[+\infty]$ respectively. If $\nu^1 > \nu^2$ system (2) has no solution. Otherwise the required interval for ν feasible to this system is $\nu^1 \leq \nu \leq \nu^2$.

Also in every iteration of this algorithm, we encounter the problem of solving a 2-variable LP in the variables α, λ of the following form:

$$\begin{aligned} &\text{Maximize } \lambda \\ \text{Subject to} \quad & c_p\alpha + d_p\lambda \geq g_p, \qquad p = 1 \text{ to } m, \\ & \alpha \text{ real, and } \lambda \geq 0 \end{aligned} \tag{3}$$

starting with a given feasible solution (α_1, λ_1). Now we give the procedure, we will call it **Subroutine 2**, for solving this 2-variable LP. Here we discuss the version of Subroutine 2 in the case where the set of feasible solutions of (3) is bounded. The version of Subroutine 2 when the set of feasible solutions of (3) may be unbounded is discussed later in Section 3.

Subroutine 2: Each step in this algorithm to solve the 2-variable LP (3) consists of three substeps. We will describe the general r-th step in this algorithm beginning with the feasible solution (α_r, λ_r).

Substep 1: Fix $\lambda = \lambda_r$, find the interval of feasibility for α with it. Let (α_{r1}, λ_r) be its midpoint. If this interval of feasibility consists of the single point (α_{r1}, λ_r) , then this point (α_{r1}, λ_r) is the optimum solution of (3), terminate. Otherwise continue.

Now fix $\alpha = \alpha_{r1}$, find λ_{r1} the maximum value of λ feasible to (3) with it using Subroutine 1. With the feasible solution $(\alpha_{r1}, \lambda_{r1})$ go to Substep 2.

Substep 2: Fix $\lambda = \lambda_{r1}$, and find the midpoint $(\alpha_{r2}, \lambda_{r1})$ of the interval of feasibility to (3) with it. If this point is the only point in this interval, then it is the optimum solution of (3), terminate. Otherwise, continue.

Now fix $\alpha = \alpha_{r2}$, find λ_{r2} , the maximum value of λ feasible to (3) with it, using Subroutine 1. With he feasible solution $(\alpha_{r2}, \lambda_{r2})$ go to Sustep 3.

Substep 3: The half-line joining the point (α_{r1}, λ_r) to $(\alpha_{r2}, \lambda_{r1})$ and continuing in the same direction has the *general point* $(\alpha_{r1}+\gamma(\alpha_{r2}-\alpha_{r1}), \lambda_r+\gamma(\lambda_{r1}-\lambda_r))$, where $\gamma \geq 0$ is a nonnegative parameter. The last point on this half-line feasible to (3) is this *general point* corresponding to $\gamma = \gamma^*$, where this γ^* is the maximum value of γ for which this *general point* is feasible to (3).

Among the pair of points $(\alpha_{r2}, \lambda_{r2}), (\alpha_{r1} + \gamma^*(\alpha_{r2} - \alpha_{r1}), \lambda_r + \gamma^*(\lambda_{r1} - \lambda_r))$ denote the one corresponding to the maximum value for the λ-coordinate in it by $(\alpha_{r+1}, \lambda_{r+1})$.

If the difference $\lambda_{r+1} - \lambda_r$ is smaller than some small tolerance, terminate the algorithm with $(\alpha_{r+1}, \lambda_{r+1})$ as a near optimum solution of the 2-variable LP (3). Otherwise, with $(\alpha_{r+1}, \lambda_{r+1})$ as the initial feasible solution go to the next step in the algorithm.

Terminate this process when the diffference between the values of λ in consecutive points in the sequence becomes $\leq \epsilon$, where ϵ is a small positive tolerance value. If (α_s, λ_s) is the final point in the sequence generated, choose the feasible point among $(\alpha_s \pm \epsilon, \lambda_s - \epsilon)$ as the near optimum solution of (3).

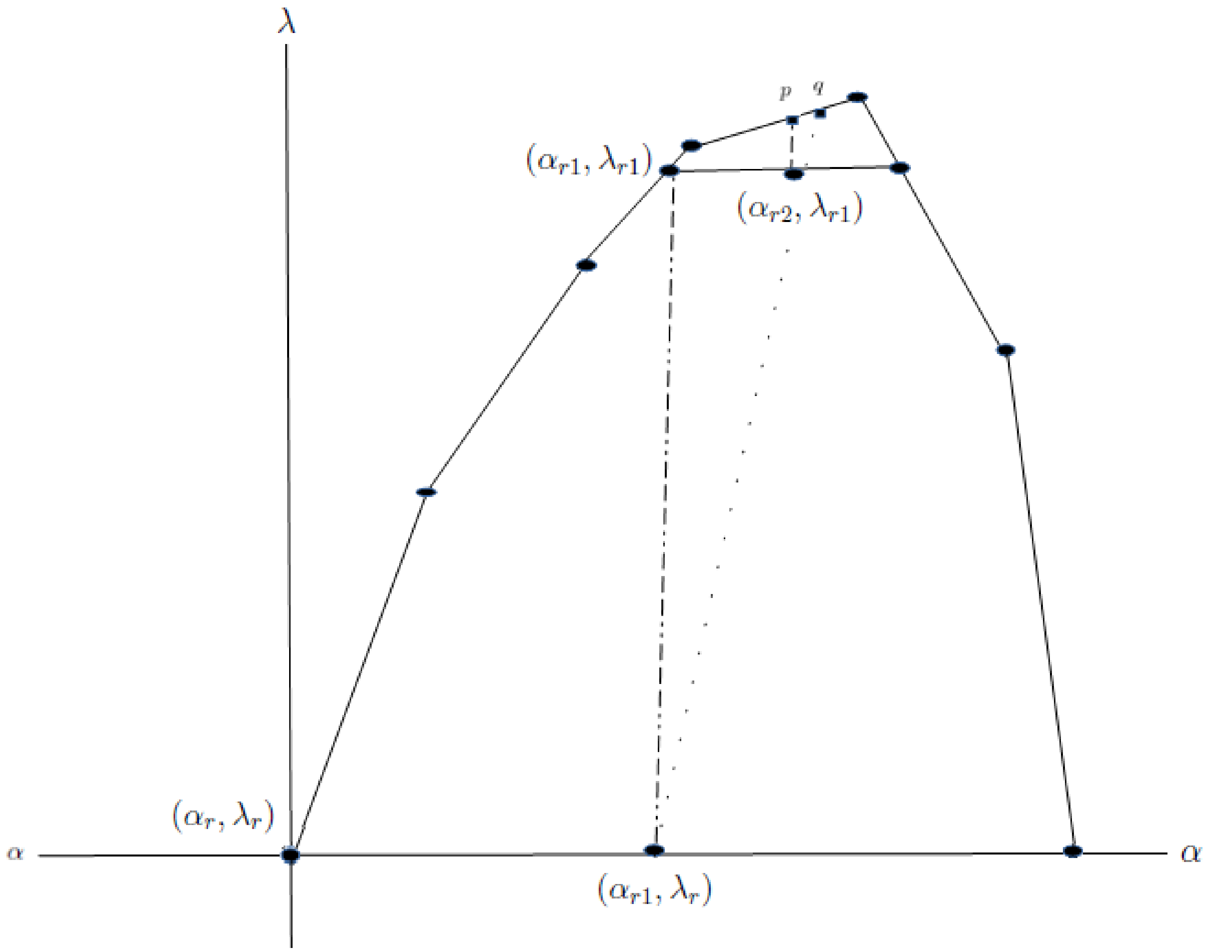

Figure 2: An illustration of the feasible region of (3). Here α is plotted on the horizontal axis, and λ is plotted along the vertical axis. Starting with the feasible solution (α_r, λ_r), the output points in Substeps 2, 3 respectively in this Step of the algorithm are $p = (\alpha_{r2}, \lambda_{r2}), q = (\alpha_{r1} + \gamma^*(\alpha_{r2} - \alpha_{r1}), \lambda_r + \gamma^*(\lambda_{r1} - \lambda_r))$. Among these two points, the one with a higher value for λ, here $q = (\alpha_{r1} + \gamma^*(\alpha_{r2} - \alpha_{r1}), \lambda_r + \gamma^*(\lambda_{r1} - \lambda_r))$ denoted by $(\alpha_{r+1}, \lambda_{r+1})$ is the beginning

point for the next step in the algorithm. Thanks to Olof Minto for his help on Figures 2, 3.

Now we will describe the general iteration in SM-7-6 in the case when K is bounded.

General iteration beginning with the initial IFS $\hat{x}$

Find $\delta(\hat{x}), T(\hat{x}), \bar{\hat{x}}$. If $\bar{\hat{x}}$ is a boundary point of K, i.e. satifies $A_{i.}\bar{\hat{x}} = b_i$ for some $i = 1$ to m, then $H(\bar{\hat{x}})$ must be the same as $\{x : A_{i.}x = b_i\}$, so $\bar{\hat{x}}$ is an optimum solution of the origional LP, terminate the algorithm with this conclusion.

Otherwise $\bar{\hat{x}}$ is an IFS of K, continue.

For each $i \in T(\hat{x})$ find using Subroutine 1, the interval of values of the parameter α satisfying

$$A_{p.}(\alpha\bar{\hat{x}} + (1-\alpha)\hat{x}^i) \geq b_p \qquad \text{for } p = 1 \text{ to } m.$$

This interval includes $0 \leq \alpha \leq 1$ since for these values of α the point $\alpha\bar{\hat{x}} + (1-\alpha)\hat{x}^i$ is on the line segment joining $\hat{x}^i$ to $\bar{\hat{x}}$. Suppose this interval is $0 \leq \alpha \leq \alpha^{i2}$. For each $i \in T(\hat{x})$, $\hat{x}^{i2} = \alpha^{i2}\bar{\hat{x}} + (1-\alpha^{i2})\hat{x}^i$ is the other boundary point of K on the straight line joining $\hat{x}^i$ to $\bar{\hat{x}}$.

Let r be the value of i attaining the minimum in minimum $\{c\hat{x}^{i2} : i \in T(\hat{x})\}$

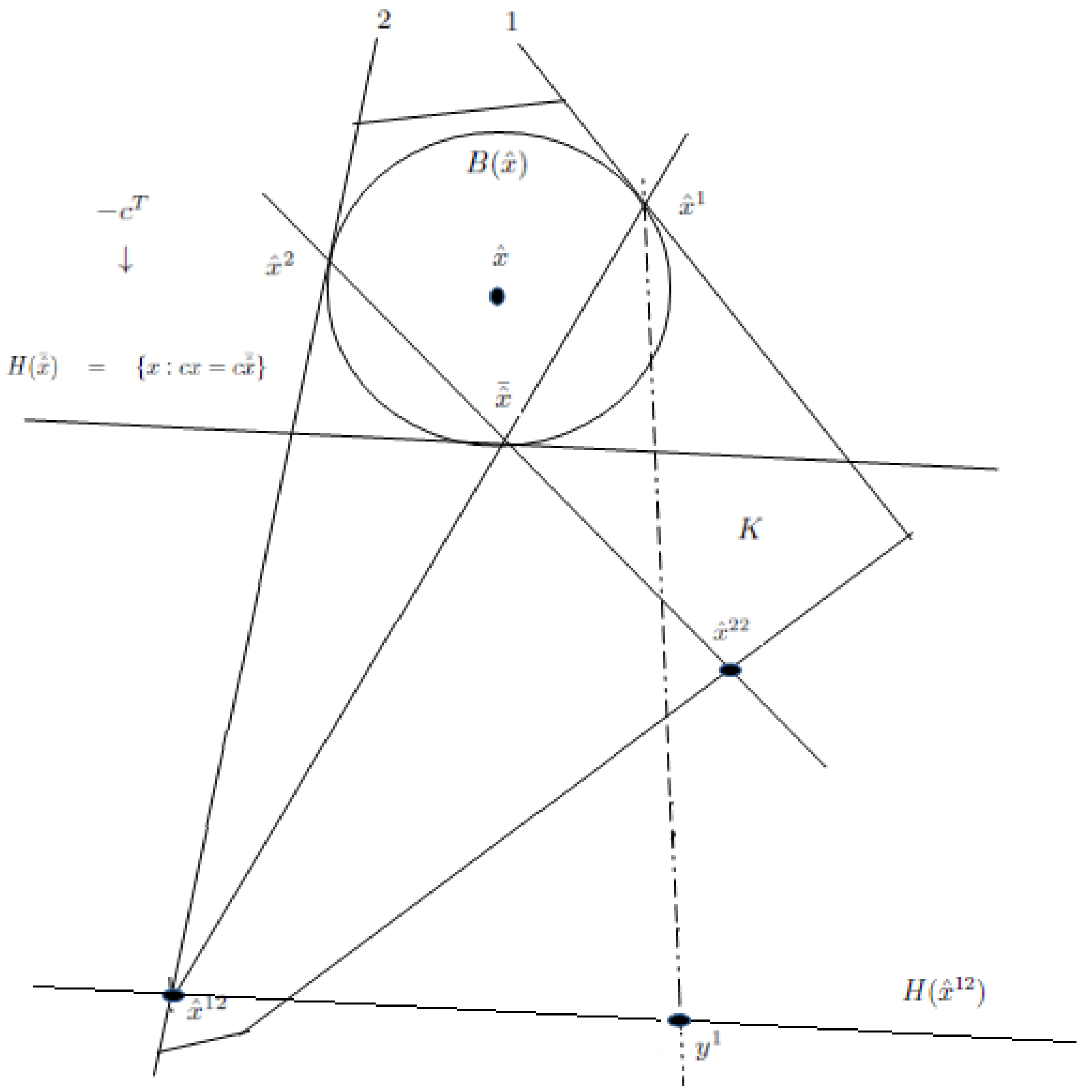

Figure 3: $\hat{x}$ is an IFS of K, $B(\hat{x})$ is the largest sphere with center $\hat{x}$ as center inside K. $\bar{\hat{x}}$ is the point in $B(\hat{x})$ with the smallest value for cx, and $H(\bar{\hat{x}})$ is the objective plane through $\bar{\hat{x}}$, it is the tangent plane to $B(\hat{x})$ at $\bar{\hat{x}}$. Facets 1, 2 of K are tangent planes to $B(\hat{x})$ with touching points $\hat{x}^1, \hat{x}^2$ respectively, so $T(\hat{x}) = \{1, 2\}$. The lines joining $\hat{x}^1, \hat{x}^2$ to $\bar{\hat{x}}$ intersect the boundary of K again at $\hat{x}^{12}, \hat{x}^{22}$ respectively. $c\hat{x}^{12} = \text{minimum}\{c\hat{x}^{12}, c\hat{x}^{22}\}$. So the index r defined above

is 1 in this example.

The objective plane through the point $\hat{x}^{r2}$ is

$H(\hat{x}^{r2}) = \{x : cx = c\hat{x}^{r2}\}$.

The orthogonal projection from $\hat{x}^r$ on $H(\hat{x}^{r2})$ is $y^1 = \hat{x}^r - \alpha_r c^T$ where α_r is determined from the equation $c(\hat{x}^r - \alpha_r c^T) = c\hat{x}^{r2}$, and hence $\alpha_r = (c\hat{x}^r - c\hat{x}^{r2})/cc^T$.

The half-line $\{x : x = \hat{x}^r - \lambda c^T, \lambda \geq 0\}$ intersects the objective plane $H(\hat{x}^{r2})$ at the point y^1.

Let $L^{22} = \{\beta\hat{x}^{r2} + (1-\beta)y^1 - \lambda c^T : \beta \text{ real}, \lambda \geq 0\}$, the 2-dimensional half-space determined by the line joining $\hat{x}^{r2}$ and y^1, and the direction $-c^T$.

In $L^{22} \cap K$, since $cy^1 = c\hat{x}^{r2}$, minimizing cx on it is the same as minimizing $(c\hat{x}^{r2} - \lambda cc^T)$, and hence this problem is equivalent to the following problem:

Maximize λ

Subject to $\quad A[\beta\hat{x}^{r2} + (1-\beta)y^1 - \lambda c^T)] \geq b$

$\lambda \geq 0$, β real

which can be solved by applying Subroutine 2. If $(\hat{\beta}, \hat{\lambda})$ is the solution of this 2-variable LP, then in terms of the original variables $x^2 = \hat{\beta}\hat{x}^r + (1-\hat{\beta})\hat{x}^{r2} - \hat{\lambda}c^T$ is the final solution obtained in this iteration. This may not be an IFS.

Let L be the straight lie through $\bar{\hat{x}}$ parallel to the straight line joining $\hat{x}^{r2}$ and y^1; and suppose L^1 and L^2 are the end points of L in K.

We now have two choices to determine a point $\hat{\hat{x}}$ on L ; the best among them has to be determined by computational tests to find out which one gives better results:

CHOICE 1: $\hat{\hat{x}}$ is the point on L intersected with K which has the maximum value of $\delta(x)$

CHOICE 2: $\hat{\hat{x}}$ is $(L^1 + L^2)/2$

Take $(x^2 + \hat{\hat{x}})/2$, the midpoint joining x^2 and $\hat{\hat{x}}$, this will be an IFS of K; and $H((x^2 + \hat{\hat{x}})/2)$ is the objective plane through that point. Add the constraint $cx \leq c((x^2 + \hat{\hat{x}})/2)$ as the $(m+1)$th constraint to the original LP. This gives the problem

$$\begin{aligned} \min \quad & z = cx && (3) \\ \text{subject to} \quad & Ax \geq b && (4) \\ \text{and} \quad & cx \leq c((x^2 + \hat{\hat{x}})/2) \end{aligned}$$

The objective plane $H((x^2 + \hat{\hat{x}})/2)$ divides R^n into two half-spaces, of which $H_{\downarrow}(x^2 + \hat{\hat{x}})/2)$ is the half-space in the direction $-c^T$. $K_{\downarrow} = K \cap H_{\downarrow}(x^2 + \hat{\hat{x}})/2)$ is the set of feasible slutions of (4); and hence the set of optimum solutions of (1) and (4) are the same.

The point $(1/2)(x^2 + (x^2 + \hat{\hat{x}})/2)$ is an IFS of $K_{\downarrow}$. With that as the initial IFS, repeat the application of this algorithm to solve the LP (4). Continue this way. When you repeat this process, the problem corresponding to (4) is just (4) with a smaller RHS constant in the $(m + 1)$th constraint of it.

The repetition process is continued until the difference between objective valus 0f cx^2 in consecutive iterations of the algorithm becomes smaller than a small ϵ. The final point corresponding to x^2 in the final repetition is taken as an approximate optimum solution of the original problem (1).

3 Changes for the General Case When an Initial IFS in K Is Not Given, and When K is Not Known to be Bounded

We now consider the general LP model in nonnegative variables. If the model being solved includes "$\leq$ " inequality constraints, $D_{i.}y \leq d_i$, replace each of them with the equivalent $-D_{i.}y \geq -d_i$.

Also, suppose the model being solved has r equality constraints

$D_{i.}y = d_i, \quad i = 1, ..., r$

Then replace these r equations in the model by r inequalities

$D_{i.}y \geq d_i, \quad i = 1, ..., r$

and then include $\bar{y} = \sum_{i=1}^{r}(D_{i.}y - d_i)$ to the objective function being minimized, with a large positive cost coefficient. In the optimum solution of the resulting model if $\bar{y}$ has a positive value, it is an indication that the original model with the equatity constrints in infeasible.

So, with these modifications we can transform the general LP model into the form (1).

1. So let us now consider the LP (1), but suppose an initial IFS for it is not available. Then we consider the modified problem:

$$\text{Minimize} \qquad z = c_1x_1 + ... + c_nx_n + c_{n+1}x_{n+1} \qquad (5)$$

$$\text{subject to} \qquad Ax + ex_{n+1} \geq b$$

$$\text{and} \qquad x_{n+1} \geq 0, \quad -x_{n+1} \geq b_{m+1},$$

where x_{n+1} is a nonnegative artificial variable, and e is the column vector in R^m with all entries $= 1$. Let $\bar{x}_{n+1} = \delta_1 +$ $maximum\{0, b_i : i = 1, ..., m\}$, and $b_{m+1} = -\bar{x}_{n+1} - \delta_2$, where δ_1, δ_2 are strictly positive numbers. c_{n+1} is a large positive cost coefficient of x_{n+1} in (5).

We will now consider (5) as the problem to solve, and continue to denote the new column vector of decision variables $x_1, ..., x_n, x_{n+1}$ by the same symbol x; its cost vector $(c_1, ..., c_n, c_{n+1})$ by the same symbol c. The column vector $\bar{x}$ with $x_1 = ... = x_n = 0$ and $x_{n+1} = \bar{x}_{n+1}$ is an IFS for (5). We can solve (5) beginning with this initial IFS by the algorithm discussed earlier, and since c_{n+1} is a large positive cost coefficient , x_{n+1} will have value 0 in its optimum solution, if the original LP (1) has an optimum solution.

2. Now consider the case in which K, the set of feasible solutions of the original LP, is not known to be bounded. In this case, since the feasibility set of the 2-variable LP of the form (3) encountered in iterations of the algorithm may be unbounded, some changes have to be made in Subroutine 2 used to solve it. Below, we discuss the modified Subroutine 2 used to solve it .

Subroutine 2 for the case when the set of feasible solutions of (3) is not known to be bounded

Let (α_1, λ_1^s) be the initial feasible solution of (3) available. Go to the new Sustep 0.

Substep 0: Fixing $\alpha = \alpha_1$ find the maximum value of λ in the set of feasible solutions for (3). If it is ∞ , then the maximum value of λ in (3) is $+\infty$, and $\{(\alpha_1, \lambda) : \lambda \geq \lambda_1^s\}$ is a feasible half-line for (3) along which λ diverges to ∞, terminate.

Otherwise, let λ_1 be the maximum value of λ in (3) when α is fixed at α_1. With the feasible solution (α_1, λ_1) as the initial feasible solution go to Substep 1. Considering the general r-th step, let (α_r, λ_r) be the initial feasible solution.

Substep 1: When we fix $\lambda = \lambda_r$ in (3), suppose the interval of feasibility for α is $\alpha_r \leq \alpha \leq \infty$. Now check how the maximum value of λ varies as a function of α as it varies from α_r to ∞. If after some value of α in this interval, say $\alpha = \bar{\alpha}_r$, λ remains constant at $\bar{\lambda}_r$, then $(\bar{\alpha}_r, \bar{\lambda}_r)$ is an optimum solution of (3). On the other hand, if the maximum value of λ in (3) keeps going up as α varies from α_r to ∞ , then λ is unbounded in (3), and you can get a half-line along which it diverges to ∞ from this.

On the other hand, with λ fixed at λ_r in (3), if the interval of feasibility for α is bounded, let (α_{r1}, λ_r) be its midpoint. If this interval of feasibilty consists of the single point (α_{r1}, λ_r), then this point (α_{r1}, λ_r) is the optimum solution of (3), terminate. Otherwise, fix $\alpha = \alpha_{r1}$, and find λ_{r1}, the maximum value of λ feasible to (3) with it, using Subroutine 1. With the feasible solution $(\alpha_{r1}, \lambda_{r1})$ go to Substep 2.

Substep 2: Fix $\lambda = \lambda_{r1}$. With it, if the interval of feasibility for α in (3) is (α_{r1}, ∞) ; check how the maximum value of λ in the feasibility interval of (3) varies as α varies in this interval. If after some value of α, say α^2_{r1}, in this interval, it remains constant at λ^2_{r1}, then $(\alpha^2_{r1}, \lambda^2_{r1})$ is an optimum solution of (3).

On the other hand, if the maximum value of λ in (3) keeps going up as α varies from α_{r1} to ∞ , then λ is unbounded in (3), and you can get a half-line along which it diverges to ∞ from this.

With λ fixed at λ_{r1} in (3) if the interval of feasbility for α is bounded, let $(\alpha_{r2}, \lambda_{r1})$ be its midpoint. If this interval of feasibilty consists of the single point $(\alpha_{r2}, \lambda_{r1})$, then this point $(\alpha_{r2}, \lambda_{r1})$ is the optimum solution of (3), terminate. Otherwise, fix $\alpha = \alpha_{r2}$ in (3), and let λ_{r2} the maximum value of λ in it. With the feasible solution $(\alpha_{r2}, \lambda_{r2})$ go to Substep 3.

Substep 3: The half-line joining the point (α_{r1}, λ_r) to $(\alpha_{r2}, \lambda_{r1})$ and continuing in the same direction has the *general point* $(\alpha_{r1} + \beta(\alpha_{r2} - \alpha_{r1}), \lambda_r + \beta(\lambda_{r1} - \lambda_r))$, where $\beta \geq 0$ is a nonnegative parameter. The last point on this half-line feasible to (3) is this *general point* corresponding to $\beta = \beta^*$, where this β^* is the maximum value of β for which this *general point* is feasible to (3).

If $\beta^* = \infty$, then the half-line $\{(\alpha_{r1} + \beta(\alpha_{r2} - \alpha_{r1}), \beta(\lambda_{r1} - \lambda_r)) : \beta \geq 0\}$ is a feasible half-line along which λ diverges to ∞.

If β^* is finite, among the pair of points $(\alpha_{r2}, \lambda_{r2}), (\alpha_{r1} + \beta^*(\alpha_{r2} - \alpha_{r1}), \lambda_r + \beta^*(\lambda_{r1} - \lambda_r))$ denote the one corresponding to the maximum value for the λ-coordinate in it by $(\alpha_{r+1}, \lambda_{r+1})$.

If the difference $\lambda_{r+1} - \lambda_r$ is smaller than some small tolerance, terminate the algorithm with $(\alpha_{r+1}, \lambda_{r+1})$ as a near optimum solution of the 2-variable LP (3). Otherwise, with $(\alpha_{r+1}, \lambda_{r+1})$ as the initial feasible solution go to the next step in the algorithm.

Terminate this process when the diffference between the values of λ in consecutive points in the sequence becomes $\leq \epsilon$, where ϵ is a small positive tolerance value. If (α_s, λ_s) is the final point in the sequence generated, choose the feasible point among $(\alpha_s \pm \epsilon, \lambda_s - \epsilon)$ as the near optimum solution of (3).

Changes in the General Iteration Beginning with the Initial IFS $\hat{x}$

The work here is carried out as described earlier.

For each $i \in T(\hat{x})$ find using Subroutine 1, the interval of values of the parameter α satisfying

$$A_{p.}(\alpha\bar{\hat{x}} + (1 - \alpha)\hat{x}^i) \geq b_p \qquad \text{for } p = 1 \text{ to } m.$$

This interval includes $0 \leq \alpha \leq 1$ since for these values of α the point $\alpha\bar{\hat{x}} + (1 - \alpha)\hat{x}^i$ is on the line segment joining $\hat{x}^i$ to $\bar{\hat{x}}$. Suppose this interval is $0 \leq \alpha \leq \alpha^{i2}$. If for some $i \in T(\hat{x})$, this $\alpha^{i2} = \infty$, then for that i, $\{\hat{x}^i + \alpha(\bar{\hat{x}} - \hat{x}^i), \alpha \geq 0\}$ is a half-line in K along which the objective function cx diverges to $-\infty$, we terminate the algorithm with this conclusion.

Otherwise, for each $i \in T(\hat{x})$, $\hat{x}^{i2} = \alpha^{i2}\bar{\hat{x}} + (1 - \alpha^{i2})\hat{x}^i$ is the other boundary point of K on the straight line joining $\hat{x}^i$ to $\bar{\hat{x}}$.

Let r be the value of i attaining the minimum in minimum $\{c\hat{x}^{i2} : i \in T(\hat{x})\}$

Now continue as before.

Summary

In each step the Classical Simplex Method for linear programming minimizes the objective function across a 1- dimensional boundary face of K, and for this work it needs to carry out a pivot step for changing the basis matrix by one column.

In this algoithm, in each step we minimize the objective function on a 2- dimensional polyhedron in the interior of K, that is the intersection of K with a 2- dimensional hyperplane; and all this work is carried out using no pivot steps at all. I believe that ths new method will produce results which are much better that those obtained under the simplex method.

Computational results

4 References

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