Sphere Method-7-6 Using No Matrix Inversions for Linear Programs (LPs)

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Abstract

Existing software implementations for solving Linear Programming (LP) models are all based on full matrix inversion operations involving every constraint in the model in every step. This linear algebra component in these systems makes it difficult to solve dense models even with moderate size, and it is also the source of accumulating roundoff errors affecting the accuracy of the output.

We present a new version of the Sphere method, SM-7-6, for LP not using any pivot steps; and computational results on it.

Key words: Linear Programming (LP), solving LPs by descent feasible methods without using matrix inversions.

1 Sphere Method, SM-7-6, for LP

In 2006, Sphere methods for LP, IPMs based on the properties of spheres (instead of ellipsoids like in other IPMs) were introduced in Murty [2006a, b]. The initial version of the sphere method also needed pivot steps for matrix inversions, but these pivot steps only involve a subset
of constraints in the original LP. After some other versions, in this paper we describe SM-7-6, not involving any pivot steps.

SMs consider LPs in the form:

\[
\begin{align*}
\min & \quad z = cx \\
\text{subject to} & \quad Ax \geq b
\end{align*}
\]

where \( A \) is an \( m \times n \) data matrix; with a known interior feasible solution \( x \) (i.e., satisfying \( Ax > b \)). LPs in any other form can be directly transformed into this form, see [Murty 2009a, b], Murty, Oskoorouchi [2010]. Here is some basic notation that we will use.

- **Notation for rows and columns of \( A \):** \( A_i, A_j \) denote the \( i^{th} \) row, and \( j^{th} \) column of \( A \). The index \( i \) has range \( 1 \leq i \leq m \), and \( j \) ranges in \( 1 \leq j \leq n \).

- **Feasible region and its interior:** \( K \) denotes the set of feasible solutions of (1), and \( K^0 = \{x : Ax > b\} \) is its interior.

- **Facetal hyperplanes, and their half-spaces containing \( K \):** \( FH_i = \{x : A_i x = b_i\} \), the \( i^{th} \)-th facetal hyperplane of \( K \) for \( i = 1 \) to \( m \). Also, \( FH_i^+ = \{x : A_i x \geq b_i\} \) is the half-space of \( FH_i \) containing \( K \).

- **IFS:** Interior feasible solution, a point \( x \in K^0 \)

- **\( \delta(x) \):** Defined for \( x \in K \), it is the radius of the largest ball inside \( K \) with \( x \) as center. From Murty [2006a, b], we know that \( \delta(x) = \min \{ \frac{A_i x - b_i}{\|A_i\|} : i = 1, \ldots, m \} \). For any point \( x \) on the boundary of \( K \), i.e., satisfying at least one of the constraints in (1) as an equation, \( \delta(x) = 0 \) by this definition.

- **Largest ball inscribed in \( K \) with a given IFS \( x \) as center:** \( B(x) = \{y : \|y - x\| \leq \delta(x)\} \) is that largest inscribed ball in \( K \) with \( x \) as its center.

- **Touching constraint index set at a given IFS \( x \):** \( T(x) \) defined for \( x \in K^0 \), is the set of all indices \( i \) satisfying: \( \frac{A_i x - b_i}{\|A_i\|} = \min \{ \frac{A_p x - b_p}{\|A_p\|} : p = 1 \text{ to } m \} = \delta(x) \). The facetal
hyperplane $FH_i = \{ x : A_i x = b_i \}$ is a tangent plane to $B(x)$ for each $i \in T(x)$, that’s why $T(x)$ is called the **index set of touching constraints in (1) defining $K$, at $x$.**

- **Touching point** $x^i$: Defined for $x \in K^0$ and $i \in T(x)$, it is the nearest point on $FH_i$ to $x$, it is the orthogonal projection $x - A_i^T (A_i x - b_i)/||A_i||^2$ of $x$ on $FH_i$. It is the point where the ball $B(x)$ touches $FH_i$ for $i \in T(x)$.

- $H(\hat{x})$: Defined for any IFS $\hat{x} \in K^0$, $H(\hat{x}) = \{ x : cx = c\hat{x} \}$ is the objective plane through $\hat{x}$.

- $\bar{\hat{x}}$: Defined for any IFS $\hat{x} \in K^0$, it is $\hat{x} - \delta(\hat{x}) c^T/||c||$ = the bottom point of $B(\hat{x})$ in the direction $-c^T$, the point where the objective plane touches $B(\hat{x})$ when it is moved down from its present position $H(\hat{x})$, in the direction $-c^T$ until it becomes a tangent plane to $B(\hat{x})$.

- $\ddot{x}$: Defined for any IFS $\hat{x} \in K^0$ and $i \in T(\hat{x})$, it is $\hat{x}^i - c^T [ (c\hat{x}^i - c\bar{x})/cc^T ] = \text{the orthogonal projection of } \hat{x}^i \text{ on } H(\hat{x})$. 


Figure 1: $\hat{x}$ is an IFS of $K$, $B(\hat{x})$ is the largest sphere with center $\hat{x}$ as center inside $K$. $\bar{\hat{x}}$ is the point in $B(\hat{x})$ with the smallest value for $cx$, and $H(\bar{\hat{x}})$ is the objective plane through $\bar{\hat{x}}$, it is the tangent plane to $B(\hat{x})$ at $\bar{\hat{x}}$. Facets 1, 2 of $K$ are tangent planes to $B(\hat{x})$ with touching points $\hat{x}_1, \hat{x}_2$ respectively, so $T(\hat{x}) = \{1, 2\}$. $\hat{x}_1$ is the orthogonal projection of $\hat{x}_1$ on $H(\bar{\hat{x}})$. Thanks to Madhusri Katta, Vijaya Katta for Figures 1, 2.

SM-7-6 is based on feasible descent steps (starting with a feasible solution, maintaining feasibility throughout, with objective value improving monotonically), but not using any pivot steps at all. The 1st iteration begins with the given IFS $\hat{x}$, all subsequent iterations begin with the best solution (by objective value) obtained in the descent steps in the previous iteration.

We discuss a simplified version of the algorithm discussed in SM-7-3. Also, to keep the discussion simple, we restrict the discussion in this paper to the case in which $K$ is a polytope,
i.e., it is bounded.

2 A General Iteration in the Sphere Method, SM-7-5, for the LP (1)

In every iteration of SM-7-6, we face a problem of finding the interval of values of a real parameter $\nu$ say, satisfying a given system of linear inequalities in the parameter. Now we give the procedure, we will call it **Subroutine 1** for computing this interval.

**Subroutine 1:** Let the system of inequalities in $\nu$ be

$$a_t + g_t \nu \geq 0, \quad t = 1, \ldots, \ell$$

In systems like this that we encounter in SM-7-5; for any $t$ if $g_t = 0$, $a_t$ will be $\geq 0$, and hence that constraint is a redundant constraint in the system. Let

$$\nu^1 = \max \{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t > 0\}$$
$$\nu^2 = \min \{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t < 0\}$$

Here define the maximum [minimum] in the empty set to be $-\infty[+\infty]$ respectively. If $\nu^1 > \nu^2$ system (2) has no solution. Otherwise the required interval for $\nu$ feasible to this system is $\nu^1 \leq \nu \leq \nu^2$.

Also in every iteration of this algorithm, we encounter the problem of solving a 2-variable LP in the variables $\alpha, \lambda$ of the following form:

Maximize $\lambda$
Subject to $c_p \alpha + d_p \lambda \geq g_p, \quad p = 1 \text{ to } m,$
$\alpha$ real, and $\lambda \geq 0$
starting with a given feasible solution \((\alpha_1, \lambda_1)\). Now we give the procedure, we will call it **Subroutine 2**, for solving this 2-variable LP.

**Subroutine 2:** Each step in this algorithm to solve the 2-variable LP (3) consists of three substeps. We will describe the general \(r\)-th step in this algorithm beginning with the feasible solution \((\alpha_r, \lambda_r)\).

**Substep 1:** Fix \(\lambda = \lambda_r\), find the interval of feasibility for \(\alpha\) with it. Let \((\alpha_{r1}, \lambda_{r1})\) be its midpoint. If this interval of feasibility consists of the single point \((\alpha_{r1}, \lambda_{r1})\), then this point \((\alpha_{r1}, \lambda_{r1})\) is the optimum solution of (3), terminate. Otherwise continue.

Now fix \(\alpha = \alpha_{r1}\), find \(\lambda_{r1}\) the maximum value of \(\lambda\) feasible to (3) with it using Subroutine 1. With the feasible solution \((\alpha_{r1}, \lambda_{r1})\) go to Substep 2.

**Substep 2:** Fix \(\lambda = \lambda_{r1}\), and find the midpoint \((\alpha_{r2}, \lambda_{r1})\) of the interval of feasibility to (3) with it. If this point is the only point in this interval, then it is the optimum solution of (3), terminate. Otherwise, continue.

Now fix \(\alpha = \alpha_{r2}\), find \(\lambda_{r2}\), the maximum value of \(\lambda\) feasible to (3) with it, using Subroutine 1. With the feasible solution \((\alpha_{r2}, \lambda_{r2})\) go to Substep 3.

**Substep 3:** The half-line joining the point \((\alpha_{r1}, \lambda_{r1})\) to \((\alpha_{r2}, \lambda_{r1})\) and continuing in the same direction has the general point \((\alpha_{r1} + \beta(\alpha_{r2} - \alpha_{r1}), \lambda_r + \beta(\lambda_{r1} - \lambda_r))\), where \(\beta \geq 0\) is a nonnegative parameter. The last point on this half-line feasible to (3) is this general point corresponding to \(\beta = \beta^*\), where this \(\beta^*\) is the maximum value of \(\beta\) for which this general point is feasible to (3).

Among the pair of points \((\alpha_{r2}, \lambda_{r2}),(\alpha_{r1} + \beta^*(\alpha_{r2} - \alpha_{r1}), \lambda_r + \beta^*(\lambda_{r1} - \lambda_r))\) denote the one corresponding to the maximum value for the \(\lambda\)-coordinate in it by \((\alpha_{r+1}, \lambda_{r+1})\).

If the difference \(\lambda_{r+1} - \lambda_r\) is smaller than some small tolerance, terminate the algorithm with \((\alpha_{r+1}, \lambda_{r+1})\) as a near optimum solution of the 2-variable LP (3). Otherwise, with \((\alpha_{r+1}, \lambda_{r+1})\) as the initial feasible solution go to the next step in the algorithm.

Terminate this process when the difference between the values of \(\lambda\) in consecutive points in the sequence becomes \(\leq \epsilon\), where \(\epsilon\) is a small positive tolerance value. If \((\alpha_s, \lambda_s)\) is the final point in the sequence generated, choose the feasible point among \((\alpha_s \pm \epsilon, \lambda_s - \epsilon)\) as the near
optimum solution of (3).

Now we will describe the general iteration in SM-7-6.

**General iteration beginning with the initial IFS \( \hat{x} \)**

Find \( \delta(\hat{x}), T(\hat{x}), \bar{x} \). If \( \bar{x} \) is a boundary point of \( K \), i.e. satisfies \( A_i \bar{x} = b_i \) for some \( i = 1 \) to \( m \), then \( H(\bar{x}) \) must be the same as \( \{ x : A_i x = b_i \} \), so \( \bar{x} \) is an optimum solution of the original LP, terminate the algorithm with this conclusion.

Otherwise \( \bar{x} \) is an IFS of \( K \), continue.

For each \( i \in T(\hat{x}) \) find using Subroutine 1, the interval of values of the parameter \( \alpha \) satisfying 
\[
A_p(\alpha \hat{x} + (1 - \alpha)\hat{x}^i) \geq b_p \quad \text{for } p = 1 \text{ to } m.
\]

This interval includes \( 0 \leq \alpha \leq 1 \) since for these values of \( \alpha \) the point \( \alpha \hat{x} + (1 - \alpha)\hat{x}^i \) is on the line segment joining \( \hat{x}^i \) to \( \bar{x} \). Suppose this interval is \( 0 \leq \alpha \leq \alpha^2 \). For each \( i \in T(\hat{x}) \), \( \hat{x}^i = \alpha^2 \hat{x} + (1 - \alpha^2)\hat{x}^i \) is the other boundary point of \( K \) on the straight line joining \( \hat{x}^i \) to \( \bar{x} \).

Let \( r \) be the value of \( i \) attaining the minimum in minimum \( \{ c \hat{x}^i : i \in T(\hat{x}) \} \)

Let \( \tilde{x} = \hat{x} + (1 - \epsilon^0)(\hat{x}^r - \hat{x}) \) be the point some positive tolerance \( \epsilon^0 \) away from \( \hat{x}^r \) along the line segment joining \( \hat{x}^r \) and \( \hat{x}^r \). Let \( \tilde{y} = \tilde{x} - \delta(\tilde{x})c^T/||c|| \), the bottom point of \( B(\tilde{x}) \) in the direction \(-c^T\).

**Figure 2:** \( \hat{x} \) is an IFS of \( K \), \( B(\hat{x}) \) is the largest sphere with center \( \hat{x} \) as center inside \( K \). \( \tilde{x} \) is the point in \( B(\hat{x}) \) with the smallest value for \( cx \), and \( H(\tilde{x}) \) is the objective plane through \( \tilde{x} \), it is the tangent plane to \( B(\hat{x}) \) at \( \tilde{x} \). Facets 1, 2 of \( K \) are tangent planes to \( B(\hat{x}) \) with touching points \( \hat{x}^1, \hat{x}^2 \) respectively, so \( T(\hat{x}) = \{ 1, 2 \} \). The lines joining \( \hat{x}^1, \hat{x}^2 \) to \( \hat{x} \) intersect the boundary of \( K \) again at \( \hat{x}^{12}, \hat{x}^{22} \) respectively. \( c\hat{x}^{12} = \text{minimum}(c\hat{x}^{12}, c\hat{x}^{22}) \). So the index \( r \) defined above is 1 in this example. The point \( \tilde{x} \) on the line joining \( \hat{x}^{12}, \hat{x}^1 \) is shown in the figure, and \( B(\tilde{x}) \) is the largest sphere with center \( \tilde{x} \) inside \( K \). \( \tilde{y} \) is the boundary point of \( B(\tilde{x}) \) with the smallest value for the objective function \( cx \) inside \( B(\tilde{x}) \).

**NOTE:** CORRECTIONS TO BE MADE IN FIGURE 2. THE POINTS \( y^1, y^2 \), THE LINE JOINING THEM, AND THE DASHED LINES THROUGH THEM ARE NOT USED IN THIS
ALGO., THEY HAVE TO BE DELETED FROM THIS FIGURE.

For each $i \in T(\hat{y})$ [NOTE: IF RESULTS ARE NOT GOOD WITH THIS, DO YOU THINK REPLACING $T(\hat{y})$ HERE WITH $\{1,\ldots,m\}$ WILL GIVE BETTER RESULTS?] find the feasibility interval of $\alpha$, with end points $\alpha_1, \alpha_2$, feasible to the following system of linear inequalities in $\alpha$:
\( A(\tilde{y} - \alpha A^T) \geq b \)
\( \alpha \text{ real variable.} \)

Here the end points \( \alpha_1, \alpha_2 \) of this interval of feasibility for this system are named so that 
\( c(\tilde{y} - \alpha_1 A^T) < c(\tilde{y} - \alpha_2 A^T). \)

The length of the line segment \( L \) joining the two points \( (\tilde{y} - \alpha_1 A^T), (\tilde{y} - \alpha_2 A^T) \) is \( ||((\tilde{y} - \alpha_1 A^T) - (\tilde{y} - \alpha_2 A^T))|| = \left|\left| \left| \left( \left( \alpha_2 - \alpha_1 \right) \right) \right| A^T \right| \right|, \) and suppose \( i = p \in T(\tilde{y}) \) maximizes this length.

[NOTE: ANOTHER WAY IS TO DEFINE \( p \) AS THE \( i \) CORRESPONDING TO \( \min \{c(\tilde{y} - \alpha_1 A^T) : i \in T(\tilde{y}) \} \). NEED TO CHECK WHETHER THIS GIVES BETTER RESULTS.]

Let \( L_p \) be the straight line containing the line segment \( S_p \). The straight line \( L_p \) and the direction \(-c^T\) together determine the 2-dimensional half-space denoted by \( L^{21} = \{\beta(\tilde{y} - \alpha_{p2} A^T_p) + (1 - \beta)(\tilde{y} - \alpha_{p1} A^T_p) + \lambda(-c^T) : \beta \text{ real}, \lambda \geq 0\} \). We will now determine the point minimizing the objective function \( cx \) in \( L^{21} \cap K \).

Then Half-line \( \{x : x = \tilde{y} - \alpha_{p2} A^T_p - \lambda c^T, \quad \lambda \geq 0\} \) intersects \( H(\tilde{y} - \alpha_{p1} A^T_p) \) in the point 
\( \tilde{y} - \alpha_{p2} A^T_p - \hat{\lambda}c^T \) where \( \hat{\lambda} \) is determined from the equation \( c(\tilde{y} - \alpha_{p2} A^T_p - \hat{\lambda}c^T) = c(\tilde{y} - \alpha_{p1} A^T_p) \).

So \( \hat{\lambda} = \{c(\tilde{y} - \alpha_{p2} A^T_p) - c(\tilde{y} - \alpha_{p1} A^T_p)\}/cc^T \)

Let \( L^{22} = \{\beta(\tilde{y} - \alpha_{p2} A^T_p - \tilde{\lambda}c^T) + (1 - \beta)(\tilde{y} - \alpha_{p1} A^T_p) - \lambda c^T : \beta \text{ real}, \lambda \geq 0\} \), the 2-dimensional half-space determined by the line joining \( (\tilde{y} - \alpha_{p2} A^T_p - \tilde{\lambda}c^T), (\tilde{y} - \alpha_{p1} A^T_p) \) and the direction \(-c^T\).

Clearly, the point minimizing \( cx \) in \( L^{21} \cap K \) is the same as the point minimizing \( cx \) in \( L^{22} \cap K \).

In \( L^{22} \cap K \) minimizing \( cx \) is the same as minimizing \( c(\tilde{y} - \alpha_{p1} A^T_p) - \lambda cc^T \). Therefore the point minimizing \( cx \) in \( L^{22} \cap K \) is the same as the optimum solution of the following problem:

Maximize \( \lambda \)
Subject to \( A[\beta((\tilde{y} - \alpha_{p2} A^T_p - \tilde{\lambda}c^T) + (1 - \beta)(\tilde{y} - \alpha_{p1} A^T_p) - \lambda c^T)] \geq b \)
\( \lambda \geq 0, \beta \text{ real} \)

which can be solved by applying Subroutine 2. If \((\hat{\beta}, \hat{\lambda})\) is the solution of this 2-variable LP, then in terms of the original variables
\( x^2 = \hat{\beta}((\tilde{y} - \alpha_{p2} A^T_p - \tilde{\lambda}c^T) + (1 - \hat{\beta})(\tilde{y} - \alpha_{p1} A^T_p) - \hat{\lambda}c^T \) is the final solution obtained in this iteration; with this as the initial IFS, the next iteration begins.
The algorithm is terminated when the objective values of $cx$ in the final solutions in consecutive iterations become $\leq$ some tolerance.

The final solution in the algorithm is accepted as a near optimum solution of the problem.

**Computational Results**
3 References


3. K. G. Murty, “Linear Equations, Inequalities, Linear Programs (LPs), and a New efficient Algorithm”, 1-36 in *Tutorials in OR*, INFORMS, 2006b.


7. K. G. Murty, Sphere Methods Using no Matrix Inversions for LPs and Extension to NLP, 0-1 ILP, on my webpage.