Sphere Method-7-6 Using No Matrix Inversions for Linear Programs(LPs)

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Abstract

Existing software implementations for solving Linear Programming (LP) models are all based on full matrix inversion operations involving every constraint in the model in every step. This **linear algebra component** in these systems makes it difficult to solve dense models even with moderate size, and it is also the source of accumulating roundoff errors affecting the accuracy of the output.

We present a new version of the Sphere method, SM- 7-6, for LP not using any pivot steps; and computational results on it.

Key words: Linear Programming (LP), solving LPs by descent feasible methods without using matrix inversions.

1 Sphere Method, SM-7-6, for LP

In 2006, Sphere methods for LP, IPMs based on the properties of spheres (instead of ellipsoids like in other IPMs) were introduced in Murty [2006a, b]. The initial version of the sphere method also needed pivot steps for matrix inversions, but these pivot steps only involve a subset

of constraints in the original LP. After some other versions, in this paper we describe SM-7-6, not involving any pivot steps.

SMs consider LPs in the form:

$$\begin{aligned} \min \quad & z = cx \\ \text{subject to} \quad & Ax \geq b \end{aligned} \tag{1}$$

where A is an $m \times n$ data matrix; with a known interior feasible solution x (i.e., satisfying $Ax > b$). LPs in any other form can be directly transformed into this form, see [Murty 2009a, b], Murty, Oskoorouchi [2010]. Here is some basic notation that we will use.

- **Notation for rows and columns of A:** $A_{i.}, A_{.j}$ denote the ith row, and j^{th} column of A. The index i has range $1 \leq i \leq m$, and j ranges in $1 \leq j \leq n$.
- **Feasible region and its interior:** K denotes the set of feasible solutions of (1), and $K^0 = \{x : Ax > b\}$ is its interior.
- **Facetal hyperplanes, and their half-spaces containing K:** $FH_i = \{x : A_{i.}x = b_i\}$, the i-th facetal hyperplance of K for $i = 1$ to m. Also, $FH_i^+ = \{x : A_{i.}x \geq b_i\}$ is the half-space of FH_i containing K.
- **IFS:** Interior feasible solution, a point $x \in K^0$
- $\delta(x)$: Defined for $x \in K$, it is the radius of the largest ball inside K with x as center. From Murty [2006a, b], we know that $\delta(x) = \text{minimum}\{\frac{A_{i.}x - b_i}{||A_{i.}||} : i = 1, ..., m\}$. For any point x on the boundary of K , i.e., satisfying at least one of the constraints in (1) as an equation, $\delta(x) = 0$ by this definition.
- **Largest ball inscribed in K with a given IFS x as center:** $B(x) = \{y : ||y - x|| \leq \delta(x)\}$ is that largest inscribed ball in K with x as its center.
- **Touching constraint index set at a given IFS:** $T(x)$ defined for $x \in K^0$, is the set of all indices i satisfying: $\frac{A_{i.}x - b_i}{||A_{i.}||} = \text{Minimum}\{\frac{A_{p.}x - b_p}{||A_{p.}||} : p = 1 \text{ to } m\} = \delta(x)$. The facetal

hyperplane $FH_i = \{x : A_{i.}x = b_i\}$ is a tangent plane to $B(x)$ for each $i \in T(x)$, that's why $T(x)$ is called the **index set of touching constraints in (1) defining** K, at x.

- **Touching point** x^i **:** Defined for $x \in K^0$ and $i \in T(x)$, it is the nearest point on FH_i to x, it is the orthogonal projection $x - A_{i.}^T(A_{i.}x - b_i)/||A_{i.}||^2$ of x on FH_i. It is the point where the ball $B(x)$ touches FH_i for $i \in T(x)$.

- $H(\hat{x})$: Defined for any feasible solution $\hat{x} \in K$, $H(\hat{x}) = \{x : cx = c\hat{x}\}$ is the objective plane through $\hat{x}$

- $\bar{\hat{x}}$: Defined for any IFS $\hat{x} \in K^0$, it is $= \hat{x} - \delta(\hat{x})c^T/||c|| =$ the bottom point of $B(\hat{x})$ in the direction $-c^T$, the point where the objective plane touches $B(\hat{x})$ when it is moved down from its present position $H(\hat{x})$, in the direction $-c^T$ until it becomes a tangent plane to $B(\hat{x})$

- $\bar{\hat{x}}^i$: Defined for any IFS $\hat{x} \in K^0$ and $i \in T(\hat{x})$, it is $= \hat{x}^i - c^T[(c\hat{x}^i - c\bar{\hat{x}})/cc^T] =$ the orthogonal projection of $\hat{x}^i$ on $H(\bar{\hat{x}})$.

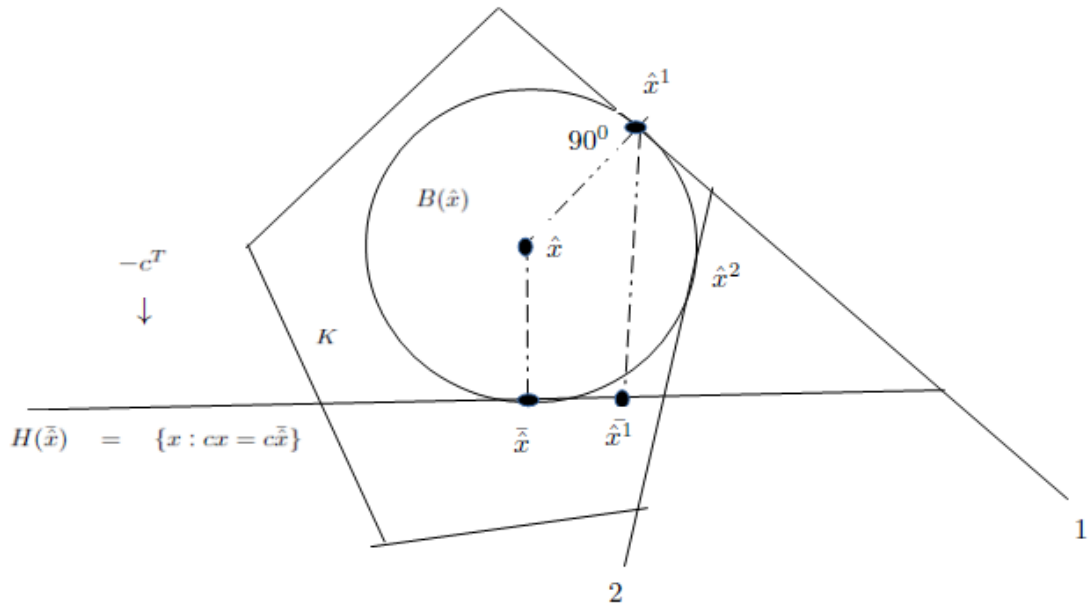


Figure 1: $\hat{x}$ is an IFS of K, $B(\hat{x})$ is the largest sphere with center $\hat{x}$ as center inside K. $\bar{\hat{x}}$ is the point in $B(\hat{x})$ with the smallest value for cx, and $H(\bar{\hat{x}})$ is the objective plane through $\bar{\hat{x}}$, it is the tangent plane to $B(\hat{x})$ at $\bar{\hat{x}}$. Facets 1, 2 of K are tangent planes to $B(\hat{x})$ with touching points $\hat{x}^1, \hat{x}^2$ respectively, so $T(\hat{x}) = \{1, 2\}$. $\bar{\hat{x}}^1$ is the orthogonal projection of $\hat{x}^1$ on $H(\bar{\hat{x}})$. Thanks to Madhusri Katta, Vijaya Katta for Figures 1, 2.

SM-7-6 is based on feasible descent steps (starting with a feasible solution, maintaining feasibility throughout, with objective value improving monotonically), but not using any pivot steps at all. The 1st iteration begins with the given IFS $\hat{x}$, all subsequent iterations begin with the best solution (by objective value) obtained in the descent steps in the previous iteration.

We discuss a simplified version of the algorithm discussed in SM-7-3. Also, to keep the discussion simple, we first restrict the discussion to the case in which K is a polytope, i.e., it is

bounded.

2 A General Iteration in the Sphere Method, SM-7-6, for the LP (1)

In every iteration of SM- 7-6, we face a problem of finding the interval of values of a real parameter ν say, satisfying a given system of linear inequalities in the parameter. Now we give the procedure, we will call it **Subroutine 1** for computing this interval.

Subroutine 1: Let the system of inequalities in ν be

$$a_t + g_t\nu \geq 0, \qquad t = 1, ..., \ell \tag{2}$$

In systems like this that we encounter in SM- 7-6; for any t if $g_t = 0$, $\quad a_t$ will be ≥ 0, and hence that constraint is a redundant constraint in the system. Let

$$\begin{aligned} \nu^1 &= \text{maximum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t > 0\} \\ \nu^2 &= \text{minimum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t < 0\} \end{aligned}$$

Here define the maximum [minimum] in the empty set to be $-\infty[+\infty]$ respectively. If $\nu^1 > \nu^2$ system (2) has no solution. Otherwise the required interval for ν feasible to this system is $\quad \nu^1 \leq \nu \leq \nu^2$.

Also in every iteration of this algorithm, we encounter the problem of solving a 2-variable LP in the variables α, λ of the following form:

$$\begin{array}{ccc} & \text{Maximize } \lambda & \\ \text{Subject to} & c_p\alpha + d_p\lambda \geq g_p, \qquad p = 1 \text{ to } m, & \\ & \alpha \text{ real, and } \lambda \geq 0 & \end{array} \tag{3}$$

starting with a given feasible solution (α_1, λ_1). Now we give the procedure, we will call it **Subroutine 2**, for solving this 2-variable LP. Here we discuss the version of Subroutine 2 in the case where the set of feasible solutions of (3) is bounded. The version of Subroutine 2 when the set of feasible solutions of (3) may be unbounded is discussed later in Section 3.

Subroutine 2: Each step in this algorithm to solve the 2-variable LP (3) consists of three substeps. We will describe the general r-th step in this algorithm beginning with the feasible solution (α_r, λ_r).

Substep 1: Fix $\lambda = \lambda_r$, find the interval of feasibility for α with it. Let (α_{r1}, λ_r) be its midpoint. If this interval of feasibility consists of the single point (α_{r1}, λ_r) , then this point (α_{r1}, λ_r) is the optimum solution of (3), terminate. Otherwise continue.

Now fix $\alpha = \alpha_{r1}$, find λ_{r1} the maximum value of λ feasible to (3) with it using Subroutine 1. With the feasible solution $(\alpha_{r1}, \lambda_{r1})$ go to Substep 2.

Substep 2: Fix $\lambda = \lambda_{r1}$, and find the midpoint $(\alpha_{r2}, \lambda_{r1})$ of the interval of feasibility to (3) with it. If this point is the only point in this interval, then it is the optimum solution of (3), terminate. Otherwise, continue.

Now fix $\alpha = \alpha_{r2}$, find λ_{r2} , the maximum value of λ feasible to (3) with it, using Subroutine 1. With he feasible solution $(\alpha_{r2}, \lambda_{r2})$ go to Sustep 3.

Substep 3: The half-line joining the point (α_{r1}, λ_r) to $(\alpha_{r2}, \lambda_{r1})$ and continuing in the same direction has the *general point* $(\alpha_{r1} + \beta(\alpha_{r2} - \alpha_{r1}), \lambda_r + \beta(\lambda_{r1} - \lambda_r))$, where $\beta \geq 0$ is a nonnegative parameter. The last point on this half-line feasible to (3) is this *general point* corresponding to $\beta = \beta^*$, where this β^* is the maximum value of β for which this *general point* is feasible to (3).

Among the pair of points $(\alpha_{r2}, \lambda_{r2}), (\alpha_{r1} + \beta^*(\alpha_{r2} - \alpha_{r1}), \lambda_r + \beta^*(\lambda_{r1} - \lambda_r))$ denote the one corresponding to the maximum value for the λ-coordinate in it by $(\alpha_{r+1}, \lambda_{r+1})$.

If the difference $\lambda_{r+1} - \lambda_r$ is smaller than some small tolerance, terminate the algorithm with $(\alpha_{r+1}, \lambda_{r+1})$ as a near optimum solution of the 2-variable LP (3). Otherwise, with $(\alpha_{r+1}, \lambda_{r+1})$ as the initial feasible solution go to the next step in the algorithm.

Terminate this process when the diffference between the values of λ in consecutive points in

the sequence becomes $\leq \epsilon$, where ϵ is a small positive tolerance value. If (α_s, λ_s) is the final point in the sequence generated, choose the feasible point among $(\alpha_s \pm \epsilon, \lambda_s - \epsilon)$ as the near optimum solution of (3).

Now we will describe the general iteration in SM-7-6 in the case when K is bounded.

General iteration beginning with the initial IFS $\hat{x}$

Find $\delta(\hat{x}), T(\hat{x}), \bar{\hat{x}}$. If $\bar{\hat{x}}$ is a boundary point of K, i.e. satifies $A_{i.}\bar{\hat{x}} = b_i$ for some $i = 1$ to m, then $H(\bar{\hat{x}})$ must be the same as $\{x : A_{i.}x = b_i\}$, so $\bar{\hat{x}}$ is an optimum solution of the origional LP, terminate the algorithm with this conclusion.

Otherwise $\bar{\hat{x}}$ is an IFS of K, continue.

For each $i \in T(\hat{x})$ find using Subroutine 1, the interval of values of the parameter α satisfying

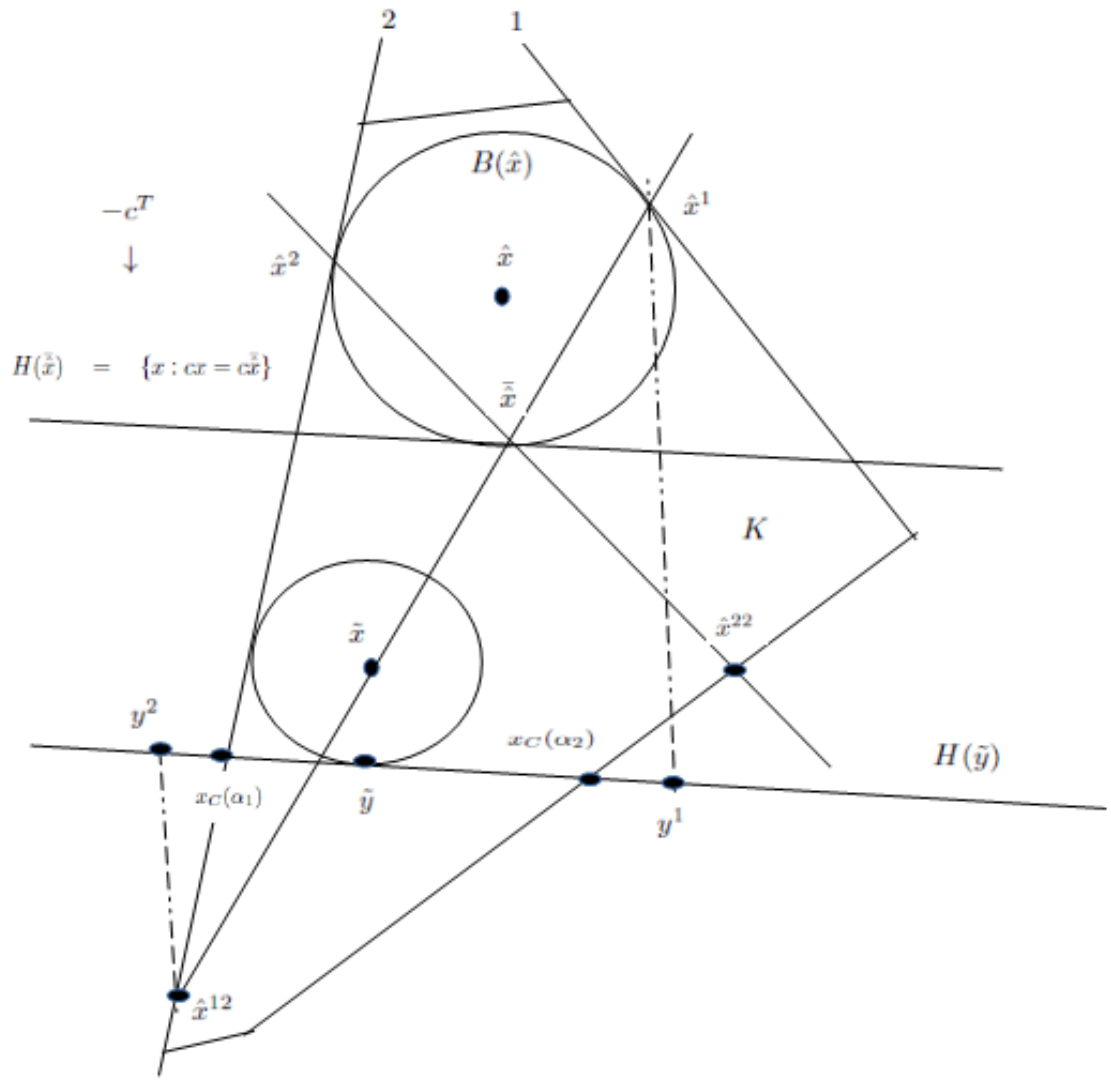
$$A_{p.}(\alpha\bar{\hat{x}} + (1-\alpha)\hat{x}^i) \geq b_p \qquad \text{for } p = 1 \text{ to } m.$$

This interval includes $0 \leq \alpha \leq 1$ since for these values of α the point $\alpha\bar{\hat{x}} + (1-\alpha)\hat{x}^i$ is on the line segment joining $\hat{x}^i$ to $\bar{\hat{x}}$. Suppose this interval is $0 \leq \alpha \leq \alpha^{i2}$. For each $i \in T(\hat{x})$, $\hat{x}^{i2} = \alpha^{i2}\bar{\hat{x}} + (1-\alpha^{i2})\hat{x}^i$ is the other boundary point of K on the straight line joining $\hat{x}^i$ to $\bar{\hat{x}}$.

Let r be the value of i attaining the minimum in minimum $\{c\hat{x}^{i2} : i \in T(\hat{x})\}$

Let $\tilde{x} = \bar{\hat{x}} + (1-\epsilon^0)(\hat{x}^{r2} - \bar{\hat{x}})$ be the point some positive tolerance ϵ^0 away from $\hat{x}^{r2}$ along the line segment joining $\hat{x}^{r2}$ and $\hat{x}^r$. Let $\tilde{y} = \tilde{x} - \delta(\tilde{x})c^T/||c||$, the bottom point of $B(\tilde{x})$ in the direction $-c^T$.

Figure 2: $\hat{x}$ is an IFS of K, $B(\hat{x})$ is the largest sphere with center $\hat{x}$ as center inside K. $\bar{\hat{x}}$ is the point in $B(\hat{x})$ with the smallest value for cx, and $H(\bar{\hat{x}})$ is the objective plane through $\bar{\hat{x}}$, it is the tangent plane to $B(\hat{x})$ at $\bar{\hat{x}}$. Facets 1, 2 of K are tangent planes to $B(\hat{x})$ with touching points $\hat{x}^1, \hat{x}^2$ respectively, so $T(\hat{x}) = \{1, 2\}$. The lines joining $\hat{x}^1, \hat{x}^2$ to $\bar{\hat{x}}$ intersect the boundary of K again at $\hat{x}^{12}, \hat{x}^{22}$ respectively. $c\hat{x}^{12} = \text{minimum}\{c\hat{x}^{12}, c\hat{x}^{22}\}$. So the index r defined above is 1 in this example. The point $\tilde{x}$ on the line joining $\hat{x}^{12}, \hat{x}^1$ is shown in the figure, and $B(\tilde{x})$ is the largest sphere with center $\tilde{x}$ inside K. $\tilde{y}$ is the boundary point of $B(\tilde{x})$ with the smallest value for the objective function cx inside $B(\tilde{x})$.



NOTE: CORRECTIONS TO BE MADE IN FIGURE 2. THE POINTS y^1, y^2 , THE LINE JOINING THEM, AND THE DASHED LINES THROUGH THEM ARE NOT USED IN THIS ALGO., THEY HAVE TO BE DELETED FROM THIS FIGURE.

For each $i \in T(\tilde{y})$ [NOTE: IF RESULTS ARE NOT GOOD WITH THIS, DO YOU THINK REPLACING $T(\tilde{y})$ HERE WITH $\{1, ..., m\}$ WILL GIVE BETTER RESULTS?] find the feasi-

bility interval of α, with end points α_{i1}, α_{i2} , feasible to the following system of linear inequalities in α:

$A(\tilde{y} - \alpha A_{i.}^T) \geq b$
α real variable.

Here the end points α_{i1}, α_{i2} of this interval of feasibility for this system are named so that $c(\tilde{y} - \alpha_{i1} A_{i.}^T) < c(\tilde{y} - \alpha_{i2} A_{i.}^T)$.

The length of the line segment S_i joining the two points $(\tilde{y} - \alpha_{i1} A_{i.}^T), (\tilde{y} - \alpha_{i2}^T A_{i.}^T)$ is $||((\tilde{y} - \alpha_{i1} A_{i.}^T) - (\tilde{y} - \alpha_{i2}^T A_{i.}^T)|| = (|(\alpha_{i2} - \alpha_{i1})|)||A_{i.}^T||$, and suppose $i = p \in T(\tilde{y})$ maximizes this length. [NOTE : ANOTHER WAY IS TO DEFINE p AS THE i CORRESPONDING TO $min\{c(\tilde{y} - \alpha_{i1} A_{i.}^T) : i \in T(\tilde{y})\}$. NEED TO CHECK WHETHER THIS GIVES BETTER RESULTS.]

Let L_p be the straight line containing the line segment S_p. The straight line L_p and the direction $-c^T$ together determine the 2-dimensional half-space denoted by $L^{21} = \{\beta(\tilde{y} - \alpha_{p2} A_{p.}^T) + (1-\beta)(\tilde{y} - \alpha_{p1} A_{p.}^T) + \lambda(-c^T) : \beta \text{ real}, \lambda \geq 0\}$. We will now determine the point minimizing the objective function cx in $L^{21} \cap K$.

Ther Half-line $\{x : x = \tilde{y} - \alpha_{p2} A_{p.}^T - \lambda c^T, \quad \lambda \geq 0\}$ intersects $H(\tilde{y} - \alpha_{p1} A_{p.}^T)$ in the point $\tilde{y} - \alpha_{p2} A_{p.}^T - \bar{\lambda} c^T$ where $\bar{\lambda}$ is determined from the equation $c(\tilde{y} - \alpha_{p2} A_{p.}^T) - \bar{\lambda} c^T) = c(\tilde{y} - \alpha_{p1} A_{p.}^T)$

So $\quad \bar{\lambda} = \{c(\tilde{y} - \alpha_{p2} A_{p.}^T) - c(\tilde{y} - \alpha_{p1} A_{p.}^T)\}/cc^T$

Let $L^{22} = \{\beta(\tilde{y} - \alpha_{p2} A_{p.}^T - \bar{\lambda} c^T) + (1-\beta)(\tilde{y} - \alpha_{p1} A_{p.}^T) - \lambda c^T : \beta \text{ real}, \lambda \geq 0\}$, the 2-dimensional half-space determined by the line joining $(\tilde{y} - \alpha_{p2} A_{p.}^T - \bar{\lambda} c^T)$ and $(\tilde{y} - \alpha_{p1} A_{p.}^T)$, and the direction $-c^T$.

Clearly, the point minimizing cx in $L^{21} \cap K$ is the same as the point minimizing cx in $L^{22} \cap K$.

In $L^{22} \cap K$ minimizing cx is the same as minimizing $c(\tilde{y} - \alpha_{p1} A_{p.}^T) - \lambda cc^T$. Therefore the point minimizing cx in $L^{22} \cap K$ is the same as the optimum solution of the following problem:

Maximize λ
Subject to $\quad A[\beta((\tilde{y} - \alpha_{p2} A_{p.}^T) - \bar{\lambda} c^T) + (1-\beta)(\tilde{y} - \alpha_{p1} A_{p.}^T) - \lambda c^T)] \geq b$
$\lambda \geq 0$, β real

which can be solved by applying Subroutine 2 to solve it. If $(\hat{\beta}, \hat{\lambda})$ is the solution of this

2-variable LP, then in terms of the original variables

$x^2 = \hat{\beta}((\tilde{y} - \alpha_{p2} A_{p.}^T) - \bar{\lambda} c^T) + (1 - \hat{\beta})(\tilde{y} - \alpha_{p1} A_{p.}^T) - \hat{\lambda} c^T$ is the final solution obtained in this iteration; with this as the initial IFS, the next iteration begins.

The algorithm is terminated when the objective values of cx in the final solutions in consecutive iterations become $\leq$ some tolerance.

The final solution in the algorithm is accepted as a near optimum solution of the problem.

Computational Results

3 Changes for the General Case When an Initial IFS in K Is Not Given, and When K is Not Known to be Bounded

1. We now consider the LP (1), but an initial IFS for it is not available. Then we consider the modified problem:

$$\begin{array}{lll} \text{Minimize} & z = c_1x_1 + ... + c_nx_n + c_{n+1}x_{n+1} & (4) \\ \text{subject to} & Ax + ex_{n+1} \geq b & \\ \text{and} & x_{n+1} \geq 0, \quad -x_{n+1} \geq b_{m+1}, & \end{array}$$

where x_{n+1} is a nonnegative artificial variable, and e is the column vector in R^m with all entries $= 1$. Let $\bar{x}_{n+1} = \delta_1 + \ maximum\{0, b_i : i = 1, ..., m\}$, and $b_{m+1} = -\bar{x}_{n+1} - \delta_2$, where δ_1, δ_2 are strictly positive numbers. c_{n+1} is a large positive cost coefficient of x_{n+1} in (4).

We will now consider (4) as the problem to solve, and continue to denote the new column vector of decision variables $x_1, ..., x_n, x_{n+1}$ by the same symbol x; its cost vector $(c_1, ..., c_n, c_{n+1})$ by the same symbol c. The column vector $\bar{x}$ with $x_1 = ... = x_n = 0$ and $x_{n+1} = \bar{x}_{n+1}$ is an IFS for (4). We can solve (4) beginning with this initial IFS by the algorithm discussed earlier, and since c_{n+1} is a large positive cost coefficient , x_{n+1} will have value 0 in its optimum solution, if the original LP (1) has an optimum solution.

2. Now consider the case in which K, the set of feasible solutions of the original LP, is not known to be bounded. In this case, since the feasibility set of the 2-variable LP of the form (3) encountered in iterations of the algorithm may be unbounded, some changes have to be made in Subroutine 2 used to solve it. Below, we discuss the modified Subroutine 2 used to solve it .

Subroutine 2 for the case when the set of feasible solutions of (3) is not known to be bounded

Let (α_1, λ_1^s) be the initial feasible solution of (3) available. Go to the new Sustep 0.

Substep 0: Fixing $\alpha = \alpha_1$ find the maximum value of λ in the set of feasible solutions for (3). If it is ∞ , then the maximum value of λ in (3) is $+\infty$, and $\{(\alpha_1, \lambda) : \lambda \geq \lambda_1^s\}$ is a feasible half-line for (3) along which λ diverges to ∞, terminate.

Otherwise, let λ_1 be the maximum value of λ in (3) when α is fixed at α_1. With the feasible

solution (α_1, λ_1) go to Substep 1. Considering the general r-th step, let (α_r, λ_r) be the initial feasible solution.

Substep 1: When we fix $\lambda = \lambda_r$ in (3), suppose the interval of feasibility for α is $\alpha_r \leq \alpha \leq \infty$. Now check how the maximum value of λ varies as a function of α as it varies from α_r to ∞. If after some value of α in this interval, say $\alpha = \bar{\alpha}_r$, λ remains constant at $\bar{\lambda}_r$, then $(\bar{\alpha}_r, \bar{\lambda}_r)$ is an optimum solution of (3). On the other hand, if the maximum value of λ in (3) keeps going up as α varies from α_r to ∞ , then λ is unbounded in (3), and you can get a half-line along which it diverges to ∞ from this.

On the other hand, with λ fixed at λ_r in (3), if the interval of feasibility for α is bounded, let (α_{r1}, λ_r) be its midpoint. Now fix $\alpha = \alpha_{r1}$, and find λ_{r1}, the maximum value of λ feasible to (3) with it, using Subroutine 1. With the feasible solution $(\alpha_{r1}, \lambda_{r1})$ go to Substep 2.

Substep 2: Fix $\lambda = \lambda_{r1}$. With it, if the interval of feasibility for α in (3) is (α_{r1}, ∞) ; check how the maximum value of λ in the feasibility interval of (3) varies as α varies in this interval. If after some value of α, say α^2_{r1}, in this interval, it remains constant at λ^2_{r1}, then $(\alpha^2_{r1}, \lambda^2_{r1})$ is an optimum solution of (3).

On the other hand, if the maximum value of λ in (3) keeps going up as α varies from α_{r1} to ∞ , then λ is unbounded in (3), and you can get a half-line along which it diverges to ∞ from this.

With λ fixed at λ_{r1} in (3) if the interval of feasbility for α is bounded, let $(\alpha_{r2}, \lambda_{r1})$ be its midpoint. Now fix $\alpha = \alpha_{r2}$ in (3), and let λ_{r2} the maximum value of λ in it. With the feasible solution $(\alpha_{r2}, \lambda_{r2})$ go to Substep 3.

Substep 3: The half-line joining the point (α_{r1}, λ_r) to $(\alpha_{r2}, \lambda_{r1})$ and continuing in the same direction has the *general point* $(\alpha_{r1} + \beta(\alpha_{r2} - \alpha_{r1}), \lambda_r + \beta(\lambda_{r1} - \lambda_r))$, where $\beta \geq 0$ is a nonnegative parameter. The last point on this half-line feasible to (3) is this *general point* corresponding to $\beta = \beta^*$, where this β^* is the maximum value of β for which this *general point* is feasible to (3).

If $\beta^* = \infty$, then the half-line $\{(\alpha_{r1} + \beta(\alpha_{r2} - \alpha_{r1}), \beta(\lambda_{r1} - \lambda_r)) : \beta \geq 0\}$ is a feasible half-line along which λ diverges to ∞.

If β^* is finite, among the pair of points $(\alpha_{r2}, \lambda_{r2}), (\alpha_{r1} + \beta^*(\alpha_{r2} - \alpha_{r1}), \lambda_r + \beta^*(\lambda_{r1} - \lambda_r))$

denote the one corresponding to the maximum value for the λ-coordinate in it by $(\alpha_{r+1}, \lambda_{r+1})$.

If the difference $\lambda_{r+1} - \lambda_r$ is smaller than some small tolerance, terminate the algorithm with $(\alpha_{r+1}, \lambda_{r+1})$ as a near optimum solution of the 2-variable LP (3). Otherwise, with $(\alpha_{r+1}, \lambda_{r+1})$ as the initial feasible solution go to the next step in the algorithm.

Terminate this process when the diffference between the values of λ in consecutive points in the sequence becomes $\leq \epsilon$, where ϵ is a small positive tolerance value. If (α_s, λ_s) is the final point in the sequence generated, choose the feasible point among $(\alpha_s \pm \epsilon, \lambda_s - \epsilon)$ as the near optimum solution of (3).

Changes in the General Iteration Beginning with the Initial IFS $\hat{x}$

The work here is carried out as described earlier until we get Minimum$\{c(\tilde{y} - \alpha_{i1} A_{i.} : i \in T(\tilde{y})\}$. If this minimum is attained by $i = p \in T(\tilde{y})$, and it is $-\infty$, then $\{\tilde{y} - \alpha A_{p.} : \alpha \geq 0\}$ is a feasible half-line along which the objective function cx diverges to $-\infty$.

Otherwise, we continue as described earlier. Again if the half-line $\{\tilde{y} - \alpha_{p2} A_{p.} - \lambda c^T : \lambda \geq \bar{\lambda}\}$ is in $L^{22} \cap K$, then cx diverges to $-\infty$ along this half-line. Otherwise we continue as described earlier.

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