

# Sphere Method-6 Using No Matrix Inversions for Linear Programs(LPs)

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## Abstract

Existing software implementations for solving Linear Programming (LP) models are all based on full matrix inversion operations involving every constraint in the model in every step. This **linear algebra component** in these systems makes it difficult to solve dense models even with moderate size, and it is also the source of accumulating roundoff errors affecting the accuracy of the output.

We present a new version of the Sphere method, SM- 6, for LP not using any pivot steps; and computational results on it.

**Key words:** Linear Programming (LP), Interior point methods (IPMs) , solving LPs by descent feasible methods without using matrix inversions.

## 1 Sphere Method, SM-6, for LP

In 2006, Sphere methods for LP, IPMs based on the properties of spheres (instead of ellipsoids like in other IPMs) were introduced in Murty [2006a, b]. The initial version of the sphere method also needed pivot steps for matrix inversions, but these pivot steps only involve a subset

of constraints in the original LP. After some other versions, in this paper we describe SM-6, not involving any pivot steps.

SMs consider LPs in the form:

$$\begin{aligned} \min \quad & z = cx \\ \text{subject to} \quad & Ax \geq b \end{aligned} \tag{1}$$

where  $A$  is an  $m \times n$  data matrix; with a known interior feasible solution  $x^1$  (i.e., satisfying  $Ax^1 > b$ ). LPs in any other form can be directly transformed into this form, see [Murty 2009a, b], Murty, Oskoorouchi [2010]. Here is some basic notation that we will use.

- **Notation for rows and columns of  $A$ :**  $A_{i.}, A_{.j}$  denote the  $i^{\text{th}}$  row, and  $j^{\text{th}}$  column of  $A$ . The index  $i$  has range  $1 \leq i \leq m$ , and  $j$  ranges in  $1 \leq j \leq n$ .
- **Feasible region and its interior:**  $K$  denotes the set of feasible solutions of (1), and  $K^0 = \{x : Ax > b\}$  is its interior.
- **Facetal hyperplanes, and their half-spaces containing  $K$ :**  $FH_i = \{x : A_{i.}x = b_i\}$ , the  $i$ -th facet hyperplane of  $K$  for  $i = 1$  to  $m$ . Also,  $FH_i^+ = \{x : A_{i.}x \geq b_i\}$  is the half-space of  $FH_i$  containing  $K$ .
- **IFS:** Interior feasible solution, a point  $x \in K^0$
- $\delta(x)$  : Defined for  $x \in K$ , it is the radius of the largest ball inside  $K$  with  $x$  as center. From Murty [2006a, b], we know that  $\delta(x) = \text{minimum}\{\frac{A_{i.}x - b_i}{\|A_{i.}\|} : i = 1, \dots, m\}$ . For any point  $x$  on the boundary of  $K$ , i.e., satisfying at least one of the constraints in (1) as an equation,  $\delta(x) = 0$  by this definition.
- **Largest ball inscribed in  $K$  with a given IFS  $x$  as center:**  $B(x) = \{y : \|y - x\| \leq \delta(x)\}$  is that largest inscribed ball in  $K$  with  $x$  as its center.
- **Touching constraint index set at a given IFS:**  $T(x)$  defined for  $x \in K^0$ , is the set of all indices  $i$  satisfying:  $\frac{A_{i.}x - b_i}{\|A_{i.}\|} = \text{Minimum}\{\frac{A_{p.}x - b_p}{\|A_{p.}\|} : p = 1 \text{ to } m\} = \delta(x)$ . The facet

hyperplane  $FH_i = \{x : A_i x = b_i\}$  is a tangent plane to  $B(x)$  for each  $i \in T(x)$ , that's why  $T(x)$  is called the **index set of touching constraints in (1) defining  $K$** , at  $x$ .

- **Touching point  $x^i$**  : Defined for  $x \in K^0$  and  $i \in T(x)$ , it is the nearest point on  $FH_i$  to  $x$ , it is the orthogonal projection  $x - A_i^T(A_i x - b_i)/\|A_i\|^2$  of  $x$  on  $FH_i$ . It is the point where the ball  $B(x)$  touches  $FH_i$  for  $i \in T(x)$ .
- **NTP (Near Touching Point) corresponding to  $i \in T(x)$**  : Defined for  $x \in K^0$  and  $i \in T(x)$ , it is the point  $(1 - \epsilon)x^i + \epsilon x$ ;  $\epsilon$  distance away from the touching point  $x^i$  on the line segment joining  $x^i$  to  $x$ , where  $\epsilon$  is a small positive tolerance.
- $H(\hat{x})$  : Defined for any IFS  $\hat{x} \in K^0$ ,  $H(\hat{x}) = \{x : cx = c\hat{x}\}$  is the objective plane through  $\hat{x}$
- $\bar{\hat{x}}$  : Defined for any IFS  $\hat{x} \in K^0$ , it is  $\hat{x} - \delta(\hat{x})c^T/\|c\|$  = the bottom point of  $B(\hat{x})$  in the direction  $-c^T$ , the point where the objective plane touches  $B(\hat{x})$  when it is moved down from its present position  $H(\hat{x})$ , in the direction  $-c^T$  until it becomes a tangent plane to  $B(\hat{x})$
- $\bar{\hat{x}}^i$  : Defined for any IFS  $\hat{x} \in K^0$ , it is  $\hat{x}^i - c^T[(c\hat{x}^i - c\bar{\hat{x}})/cc^T]$  = the orthogonal projection of  $\hat{x}^i$  on  $H(\bar{\hat{x}})$ .
- $c^i$  : For  $i = 1$  to  $m$ ,  $c^i = c^T - A_i^T[(A_i c^T)/(A_i A_i^T)]$ , the orthogonal projection of  $c^T$  on  $\{x : A_i x = 0\}$ .
- **The set  $\Gamma$**  : This set is used to store the output points along with the value for the objective function  $cx$  at them, from various descent steps in each iteration; with  $\Gamma = \emptyset$  at the beginning of the iteration. At the end of the descent cycle, the best point in  $\Gamma$  by objective value is taken as the output point in the iteration.

SM-6 is based on feasible descent steps (starting with a feasible solution, maintaining feasibility throughout, with objective value improving monotonically), but not using any pivot steps at all. The 1st iteration begins with the given IFS  $x^1$ , all subsequent iterations begin with the best solution (by objective value) obtained in the descent steps in the previous iteration.

Each iteration of this method consists of 2 steps, a Centering step, followed by a descent cycle consisting of several descent steps.

## 2 A General Iteration in the Sphere Method, SM-6, for the LP (1)

In every iteration of both SM- 6, we face a problem of finding the interval of values of a real parameter  $\gamma$  say, satisfying a given system of linear inequalities in the parameter. Now we give the procedure, we will call it **Subroutine 1** for computing this interval.

**Subroutine 1:** Let the system of inequalities in  $\gamma$  be

$$a_t + g_t\gamma \geq 0, \quad t = 1, \dots, \ell \quad (2)$$

In systems like this that we encounter in SM- 6; for any  $t$  if  $g_t = 0$ ,  $a_t$  will be  $\geq 0$ , and hence that constraint is a redundant constraint in the system. Let

$$\begin{aligned} \gamma^1 &= \text{maximum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t > 0\} \\ \gamma^2 &= \text{minimum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t < 0\} \end{aligned}$$

Here define the maximum [minimum] in the empty set to be  $-\infty$ [ $+\infty$ ] respectively. If  $\gamma^1 > \gamma^2$  system (4) has no solution. Otherwise the required interval for  $\gamma$  feasible to this system is  $\gamma^1 \leq \gamma \leq \gamma^2$ .

Now we will describe the general iteration in this method beginning with an initial IFS  $\hat{x}$ .

### 2.1 Centering step

It consists of 2, 3 repetitions of the following substep:

**Substep:** Find  $\delta(\hat{x})$ ,  $T(\hat{x})$ ,  $\bar{\hat{x}}$ , and  $\bar{\hat{x}}^i$  for each  $i \in T(\hat{x})$ . There are several possibilities which we describe below.

**Possibility 1:** If  $\bar{\hat{x}}$  is a boundary point of  $K$ , i.e. satisfies  $A_i \bar{\hat{x}} = b_i$  for some  $i = 1$  to  $m$ , then  $H(\bar{\hat{x}})$  must be the same as  $\{x : A_i x = b_i\}$ , so  $\bar{\hat{x}}$  is an optimum solution of the original LP, terminate the algorithm with this conclusion. Otherwise  $\bar{\hat{x}}$  is an IFS of  $K$ , continue.

**Possibility 2:** If  $\bar{\hat{x}} = \bar{\hat{x}}^i$  for some  $i \in T(\hat{x})$ . If this occurs, carry out a descent step from  $\hat{x}$  in the direction  $-c^T$ , i.e., find the maximum value of  $\alpha$  satisfying the system of linear inequalities:  $A(\hat{x} + \alpha(-c^T)) \geq b$ . If this maximum value is  $+\infty$ ,  $\{\hat{x} + \alpha(-c^T) : \alpha \geq 0\}$  is a feasible half-line along which  $cx$  diverges to  $-\infty$ , terminate the algorithm with it. If this maximum value is  $\alpha_1$  finite, then terminate this iteration and with  $\hat{x} + (\alpha_1 - \epsilon)(-c^T)$  for some positive tolerance  $\epsilon$  as the new IFS go to the next iteration. Otherwise  $\bar{\hat{x}} \neq \bar{\hat{x}}^i$  for all  $i \in T(\hat{x})$ , continue.

**Possibility 3:** In this Section 2.1, for  $i \in T(\hat{x})$  and  $\gamma \in R^1$ , define:

$$x_C^i(\gamma) = \bar{\hat{x}} + \gamma(\bar{\hat{x}}^i - \bar{\hat{x}})$$

and let  $L_i$  be the straight line  $= \{x_C^i(\gamma) : \gamma \in R^1\}$  joining  $\bar{\hat{x}}$  and  $\bar{\hat{x}}^i$ .

Find  $L_i \cap K = \{x_C^i(\gamma) : \gamma_{i1} \leq \gamma \leq \gamma_{i2}\}$ , where  $\gamma_{i1}$  and  $\gamma_{i2}$  are the minimum and maximum values of the parameter  $\gamma$  feasible to the system of linear inequalities  $A_i x_C^i(\gamma) \geq b_i$  for  $i = 1$  to  $m$ . This  $\gamma_{i1}, \gamma_{i2}$  can be computed by solving this system of inequalities in  $\gamma$  using Subroutine 1. We know that  $x_C^i(\gamma = 0) = \bar{\hat{x}}$ , an IFS of  $K$ , so  $\gamma_{i1} < 0$  and  $\gamma_{i2} > 0$ .

If either  $\gamma_{i1} = -\infty$  or  $\gamma_{i2} = +\infty$ , for some  $i \in T(\hat{x})$ , select  $r$  as one of these indices  $i$ . Otherwise if both  $\gamma_{i1} < \gamma_{i2}$  are finite, let  $r$  be an index  $i \in T(\hat{x})$  that maximizes  $\|x_C^i(\gamma_{i1}) - x_C^i(\gamma_{i2})\|$ .

We will now solve the line search problem: maximize  $\delta(x_C^r(\gamma))$  over the interval  $\gamma_{i1} \leq \gamma \leq \gamma_{i2}$ . There are two different approaches for solving it, which we describe below.

**Approach 1: By solving a 2-variable LP:** We verify that  $A_i x_C^r(\gamma) = A_i \bar{\hat{x}} + \gamma A_i (\bar{\hat{x}}^i - \bar{\hat{x}})$ ,

and from the definition of  $\delta(x)$  for  $x \in K$ , we can see that this problem is the same as the 2-variable LP in the variables  $\delta, \gamma$ :

$$\begin{aligned} & \max \quad \delta \\ & \text{subject to} \quad \delta \|A_i\| - \gamma [A_i(\bar{x} - \hat{x})] \leq A_i \bar{x} - b_i \quad i = 1, \dots, m \quad (3) \\ & \text{and } \delta \geq 0, \quad \gamma_{i1} \leq \gamma \leq \gamma_{i2} \end{aligned}$$

The maximum value of the variable  $\delta$  in this 2-variable LP is the maximum value of  $\delta(x_C^r(\gamma))$  over the interval  $\gamma_{i1} \leq \gamma \leq \gamma_{i2}$ .

**Approach 2: Using a Line Search algorithm:** Using the formula  $\delta(x_C^r(\gamma)) = \text{Minimum} \{(A_i x_C^r(\gamma) - b_i) / \|A_i\| : i = 1 \text{ to } m\}$  to compute  $\delta(x_C^r(\gamma))$  for any value of  $\gamma$  in the interval  $\gamma_{i1} \leq \gamma \leq \gamma_{i2}$ , we can use a popular line search algorithm like “Quadratic interpolation” (also known as “Quadratic fit line search method” ) in Nonlinear Programming for solving this problem of maximizing  $\delta(x_C^r(\gamma))$  over the interval  $\gamma_{i1} \leq \gamma \leq \gamma_{i2}$  (for example see Pages 558-560 in [1]). Let  $\bar{\gamma}$  be the optimum value for  $\gamma$  obtained for this problem. Then the output point of this substep is  $\bar{x}^C = x_C^r(\bar{\gamma})$ .

If the maximum value of  $\delta$  in this problem is  $+\infty$ , then clearly  $cx$  is unbounded below on  $K$ , terminate the algorithm with this conclusion. Otherwise the maximum value of  $\delta$  is finite, continue.

Now repeat this substep with  $\bar{x}^C$  as the initial IFS instead of  $\hat{x}$ . It can be repeated a few more times like this, as long as the  $\delta$ -value at the output point keeps increasing. The output point at the end of the final repetition of this substep, denoted by  $\bar{\bar{x}}^C$ , is called the **Center** in this iteration in SM-6. With it go to the Descent Steps.

## 2.2 Descent Steps:

For simplicity, in this step we will denote the center obtained in this iteration by the symbol  $\bar{x}$ . In keeping with the notation developed in Section 1,  $\bar{x}^i$  denotes the orthogonal projection

of  $\bar{x}$  on  $FH_i$  for  $i \in T(\bar{x})$ ,  $\bar{x}$  the bottom point of  $B(\bar{x})$  in the direction  $-c^T$ ,  $\bar{x}^i$  the orthogonal projection of  $\bar{x}^i$  on  $H(\bar{x})$ . If  $\bar{x}$  is a boundary point of  $K$ , i.e. satisfies  $A_i \bar{x} = b_i$  for some  $i = 1$  to  $m$ , then  $H(\bar{x})$  must be the same as  $\{x : A_i x = b_i\}$ , so  $\bar{x}$  is an optimum solution of the original LP, terminate the algorithm with this conclusion. Otherwise  $\bar{x}$  is an IFS of  $K$ , continue.

In this method, descent steps are taken both from IFSs of  $K$ , and also from some points outside of  $K$ ; and the output point of each descent step will be an IFS of  $K$ . Consider a descent step from a point  $x^2$  in a descent direction  $d$ , i.e., a direction satisfying  $cd < 0$ .

If  $x^2$  is an IFS of  $K$ , the maximum possible step length  $\alpha^2$  from  $x^2$  in the direction  $d$  inside  $K$  is the maximum value of  $\alpha$  satisfying  $A_i(x^2 + \alpha d) \geq b_i$  for all  $i = 1$  to  $m$ ; which is  $\text{minimum}\{(b_i - A_i x^2)/(A_i d) : \text{over all } i = 1 \text{ to } m \text{ satisfying } A_i d < 0\}$ .

If  $\{i : 1 \leq i \leq m : A_i d < 0\} = \emptyset$ , then this step length  $\alpha^2 = +\infty$ ; we terminate the algorithm with the conclusion that the objective function  $cx$  in (1) is unbounded below on  $K$ , with  $\{x^2 + \alpha d : \alpha \geq 0\}$  providing a feasible half-line along which  $cx$  diverges to  $-\infty$ . If  $\{i : 1 \leq i \leq m : A_i d < 0\} \neq \emptyset$ , Then the maximum step length is  $\alpha^2$  defined above. We take the actual step length to be  $\gamma^2 = \alpha^2 - \epsilon$ , where  $\epsilon$  is a small positive tolerance, leading to the output point  $x^2 + \gamma^2 d$ , with its objective value of  $c(x^2 + \gamma^2 d)$ .

Now consider the case in which  $x^2$  is not an IFS of  $K$ , and may be even a point outside of  $K$ . Let  $M = \{\hat{x} + \alpha d : \alpha \geq 0\}$ . In this case we need to make sure that  $M$  intersects  $K$  in its interior, as otherwise this descent step cannot produce a desired output point no matter what the step length is. For this the following conditions must hold.

**Condition 1:** If there is an  $i \leq i \leq m$  satisfying  $A_i d = 0$ , we must have  $A_i x^2 - b_i > 0$ .

**Condition 2:**  $\text{Min}\{(b_i - A_i x^2)/(A_i d) : \text{over all } i \text{ satisfying } A_i d < 0\} > \text{Max}\{(b_i - A_i x^2)/(A_i d) : \text{over all } i \text{ satisfying } A_i d > 0\}$ . This can be seen as a requirement from Subroutine 1.

In SM- 6, the point  $x^2$  from which the descent steps is taken, and the descent direction  $d$

will both be selected so that both these conditions will hold.

In this case if  $\{i : 1 \leq i \leq m \text{ and } A_i d < 0\} = \emptyset$ , the maximum step length in this descent step is  $+\infty$ , and hence the original objective function  $cx$  in (1) is unbounded below on its set of feasible solutions; and  $x^2 + \alpha d : \alpha \geq \max\{0, (b_i - A_i x^2)/(A_i d) : \text{over all } i \text{ satisfying } A_i d > 0\}$  provides a feasible half-line along which  $cx$  diverges to  $-\infty$  in  $K$ , terminate the algorithm with this conclusion.

Otherwise, take the step length to be  $\gamma^2 = \min\{(b_i - A_i x^2)/(A_i d) : \text{over all } i \text{ satisfying } A_i d < 0\} - \epsilon$ . So, the output point of this descent step in this case is  $x^2 + \gamma^2 d$  with the objective value at it  $= c(x^2 + \gamma^2 d)$ .

### Descent Steps in SM-6

From the center  $\bar{x}$  obtained in this iteration, several descent steps are carried out, and the output point in each of these descent steps is stored along with the objective value at it in a set  $\Gamma$  (which is initially  $\emptyset$ ) set up to collect all the output points of descent steps carried out in this iteration. Here are some of the possible descent steps, the most productive among them need to be determined through computational tests.

**(a).** From the center  $\bar{x}$  take descent steps in the directions  $-c^T$ , average of  $-c^i$  for  $i \in T(\bar{x})$ , average of vectors in the set  $\{A_i^T : i \in T(\bar{x}) \text{ and satisfying } cA_i^T < 0\} \cup \{-A_i^T : i \in T(\bar{x}) \text{ and satisfying } cA_i^T > 0\}$ , and (current center  $\bar{x}$ )  $-(\text{center obtained in the previous iteration})$  [this direction called the direction of the path of centers being generated, is only used from iteration 2 onwards].

**(b). Descent Steps D5.1:** For each  $i \in T(\bar{x})$  the point  $(1 - \epsilon)\bar{x}^i + \epsilon\bar{x}$ ,  $\epsilon$  distance away from the touching point  $\bar{x}^i$  on the line segment joining it to  $\bar{x}$ , is known as the NTP (Near Touching Point) corresponding to that index  $i \in T(\bar{x})$ . For each  $i \in T(\bar{x})$ , take a descent step from the NTP corresponding to it in the direction  $-c^i$ .

**(c). Descent steps D5.7:** For this step in this iteration define for parameter  $\gamma \in R^1$

$$x^i(\gamma) = \bar{x}^i + \gamma(\bar{x} - \bar{x}^i)$$

For each  $i \in T(\bar{x})$ , let us, as in the Substep discussed in Section 2.1, denote the line segment joining  $\bar{x}$  and  $\bar{x}^i$  contained on the objective plane  $H(\bar{x})$  by  $L_i = \{x^i(\gamma) : \gamma \in R^1\}$ , in parametric representation with parameter  $\gamma$ .

Since  $x^i(\gamma = 1) = \bar{x}^i$ , an interior point of  $K$ ,  $L_i$  passes through the interior of  $K$ , it must intersect  $K$  at two boundary points of  $K$  if  $L_i \cap K$  is a line segment. To find those boundary points of  $L_i \cap K$  we need to solve the following system of linear inequalities in the parameter  $\gamma$ :

$$A_p x^i(\gamma) - b_p = A_p \bar{x}^i - b_p + \gamma A_p (\bar{x} - \bar{x}^i) \geq 0 \text{ for } p = 1 \text{ to } m.$$

This system can be solved using Subroutine 1. Let  $\gamma_{i1} \leq \gamma \leq \gamma_{i2}$  be the values of the parameter  $\gamma$  feasible to this system. There are two cases to consider here.

**Case 1:** Either  $\gamma_{i1} = -\infty$ , or  $\gamma_{i2} = +\infty$ :  $\gamma_{i1}$  will be  $-\infty$  [  $\gamma_{i2}$  will be  $+\infty$  ] if  $A_p (\bar{x} - \bar{x}^i) \leq 0$  [  $A_p (\bar{x} - \bar{x}^i) \geq 0$  ] for all  $p = 1$  to  $m$ . In both instances consider the case in which all the inequalities hold as strict inequalities.

If  $A_p (\bar{x} - \bar{x}^i) < 0$  for all  $p = 1$  to  $m$ , then from the definition of  $\delta(x^i(\gamma))$ , it can be verified that it diverges to  $+\infty$  as  $\lambda \rightarrow -\infty$ . And then the point  $x^i(\gamma) - \delta(x^i(\gamma))c^T / \|c\|$  is feasible for all  $\lambda \rightarrow -\infty$  and the objective value at it diverges to  $-\infty$ . In fact if  $\beta$  is a small positive number  $< \min\{-A_p (\bar{x} - \bar{x}^i) : p = 1 \text{ to } m\}$ , then the growth rate in  $\delta(x^i(\gamma))$  as  $\gamma$  decreases is going to be larger than  $\beta$ ; and the half-line  $\{\bar{x}^i + [(\bar{x} - \bar{x}^i) + \beta(-c^T)]\gamma : \gamma \leq \gamma_{i2}\}$  is feasible and the objective value diverges to  $-\infty$  along it.

If  $\gamma_{i2} = +\infty$  and  $A_p (\bar{x} - \bar{x}^i) > 0$  for all  $p = 1$  to  $m$ ; using the same argument with the change of “ $\gamma$  negative” by “ $\gamma$  positive”, it can be verified that the same conclusion holds. So, we terminate the algorithm with this conclusion in this case.

If there is a  $p$  between 1 to  $m$  satisfying  $A_p (\bar{x} - \bar{x}^i) = 0$ , when  $\gamma_{i1} = -\infty$  [  $\gamma_{i2} = +\infty$  ] for some  $\gamma < 0$  [  $\gamma > 1$  ] carry out the descent steps in **(a)**, **(b)** from  $x^i(\gamma)$ , store the output points in the set  $\Gamma$ . Terminate this iteration, and with the best point in  $\Gamma$  by objective value at this

stage, as the initial IFS, go to the next iteration.

**Case 2:** Suppose  $\gamma_{i1} < \gamma_{i2}$  are both finite. For simplicity let us denote  $x^i(\gamma_{i1}), x^i(\gamma_{i2})$  by  $x^{i1}, x^{i2}$  respectively. So in this case  $L_i \cap K$  is the line segment  $[x^{i1}, x^{i2}]$  joining  $x^{i1}, x^{i2}$ . Define

$$x^{0i}(\alpha) = x^{i1} + \alpha(x^{i2} - x^{i1})$$

Therefore  $L_i = \{x^{0i}(\alpha) : \alpha \in R^1\}$  and  $L_i \cap K = \{x^{0i}(\alpha) : 0 \leq \alpha \leq 1\}$ . When  $\alpha \notin [0, 1]$ , even though the point  $x^{0i}(\alpha)$  is outside of  $K$ , it is possible that a descent step taken from that  $x^{0i}(\alpha)$  in the direction  $-c^T$  leads to an output point in  $K$ . The condition to be satisfied for this is that  $x^{0i}(\alpha) + \lambda(-c^T)$  is in  $K$  for some  $\lambda \geq 0$ . Denote the interval of values of  $\alpha$  satisfying this property by  $\alpha_{i1} \leq \alpha \leq \alpha_{i2}$ . We know that this interval includes  $[0, 1]$ , so  $\alpha_{i1} \leq 0, \alpha_{i2} \geq 1$ .

There are three subcases to consider here. Here the subscripts  $p, t$  are both  $\geq 1$ , and  $\leq m$ .

**Subcase 1:**  $I_1 = \{p : A_p(-c^T) > 0\} = \emptyset$ . So  $A_p(-c^T) \leq 0$  for all  $1 \leq p \leq m$ . In this subcase for each  $p$ , the quantity  $A_p(x^{0i}(\alpha) + \lambda(-c^T))$  keeps decreasing as  $\lambda$  increases from 0. So, for this quantity to be  $\geq b_i$  for some  $\lambda \geq 0$ , it must be  $\geq b_i$  for  $\lambda = 0$ . So we need  $A_p x^{0i}(\alpha) - b_p \geq 0$  for all  $1 \leq p \leq m$ . So in this case the interval is  $\alpha_{i1} = 0 \leq \alpha \leq \alpha_{i2} = 1$ .

**Subcase 2:**  $I_2 = \{p : A_p(-c^T) < 0\} = \emptyset$ . So,  $A_p(-c^T) \geq 0$  for  $1 \leq p \leq m$ . For any  $0 \leq \alpha \leq 1$  the descent step from  $x^{0i}(\alpha) \in K$  in the direction  $-c^T$  has step length  $= +\infty$  in this case, since  $A_p(x^{0i}(\alpha) - b_p + \lambda A_p(-c^T)) \geq 0$  for all  $\lambda \geq 0, 1 \leq p \leq m$  in this case.

So for any  $0 \leq \alpha \leq 1, \{x^{0i}(\alpha) + \lambda(-c^T) : \lambda \geq 0\}$  is a feasible half-line along which the objective function  $cx$  in (1) diverges to  $-\infty$ . So, we terminate the algorithm with this conclusion in this subcase.

**Subcase 3:** Both  $I_1, I_2$  are nonempty. In this Subcase it can be seen that both  $\alpha_{i1} \leq 0, \alpha_{i2} \geq 1$  are finite, and from Subroutine 1, for a fixed value of  $\alpha$ , for the system of inequalities

$$A_p(x^{0i}(\alpha) + \lambda(-c^T)) - b_p = [A_p x^{i1} - b_p + \alpha(A_p(x^{i2} - x^{i1}))] + \lambda(A_p(-c^T)) \geq 0,$$

for all  $1 \leq p \leq m$ ; to have a solution in  $\lambda$  for  $\lambda$  real; the condition to be satisfied is

$$\begin{aligned} \text{Max}\{&-[A_p x^{i1} - b_p + \alpha(A_p(x^{i2} - x^{i1}))]/(A_p(-c^T)) : \text{over all } p \in I_1\} \leq \\ \text{Min}\{&-[A_t x^{i1} - b_t + \alpha(A_t(x^{i2} - x^{i1}))]/(A_t(-c^T)) : \text{over all } t \in I_2\}. \end{aligned}$$

Further since we need this system to have a solution in  $\lambda \geq 0$ , we need in addition to the above, the following condition also to hold

$$\text{Min}\{-[A_t x^{i1} - b_t + \alpha(A_t(x^{i2} - x^{i1}))]/(A_t(-c^T)) : \text{over all } t \in I_2\} \geq 0.$$

Therefore  $\{x^{0i}(\alpha) + \lambda(-c^T) : \lambda \geq 0\} \cap \underline{K} \neq \emptyset$ , iff  $\alpha$  satisfies the following system of inequalities:

$$\begin{aligned} &-[A_p x^{i1} - b_p + \alpha(A_p(x^{i2} - x^{i1}))]/(A_p(-c^T)) + [A_t x^{i1} - b_t + \alpha(A_t(x^{i2} - x^{i1}))]/(A_t(-c^T)) \\ \leq & 0 \quad \text{for all } p \in I_1, \quad t \in I_2, \text{ and} \end{aligned}$$

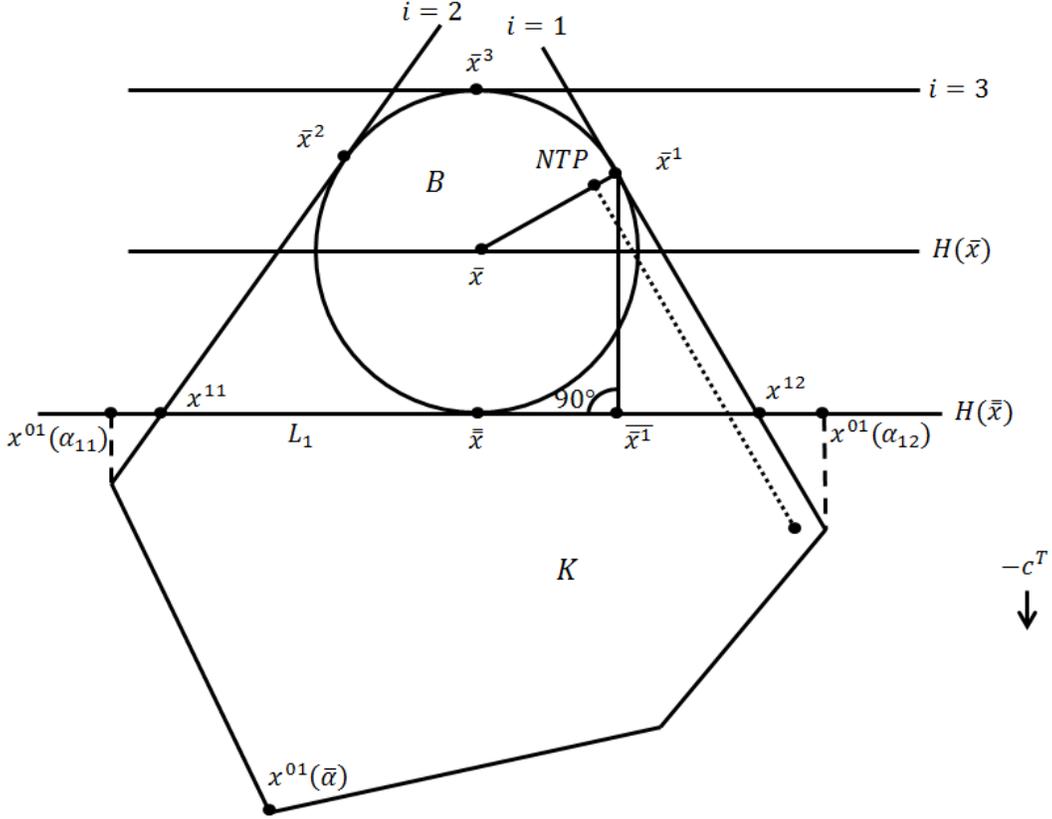
$$-[A_t x^{i1} - b_t + \alpha(A_t(x^{i2} - x^{i1}))]/(A_t(-c^T)) \geq 0 \quad \text{for all } t \in I_2.$$

This is a system of linear inequalities in the parameter  $\alpha$ , and we know that this system is feasible for all  $0 \leq \alpha \leq 1$ ; using Subroutine 1 find the complete interval of values of  $\alpha$ ;  $\alpha_{i1} \leq \alpha \leq \alpha_{i2}$  feasible to this system. We know that  $\alpha_{i1} \leq 0$ , and  $\alpha_{i2} \geq 1$ ; and both are finite.

For  $\alpha$  in this interval, when a descent step is taken from  $x^{0i}(\alpha)$  in the direction  $-c^T$  in  $K$ , the maximum step length is

$$\lambda(\alpha) = -\epsilon + \min\{(b_p - A_p x^{0i}(\alpha))/(A_p(-c^T)) : \text{over } 1 \leq p \leq m \text{ satisfying } A_p(-c^T) < 0\}.$$

and the output point from this descent step is  $x^{0i}(\alpha) + \lambda(\alpha)(-c^T)$ . The objective value at this output point is:  $f^i(\alpha) = c[x^{0i}(\alpha) + \lambda(\alpha)(-c^T)]$ . Clearly this  $f^i(\alpha)$  defined over the interval  $\alpha_{i1} \leq \alpha \leq \alpha_{i2}$  is piecewise linear convex.



**Figure 1:** The Ball  $B$  with center  $\bar{x}$ , the largest ball inside  $K$  with this center, has 3 touching facets numbered  $i = 1, 2, 3$  with touching points  $\bar{x}^1, \bar{x}^2, \bar{x}^3$  respectively. The dotted line beginning with the NTP corresponding to  $i = 1$ , is the descent step D5.1 from it, and the point at the other end of this line is the output point from this step. For descent steps D5.7, The objective plane  $H(\bar{x})$  is moved parallel to itself in the direction  $-c^T$  until it becomes a tangent plane to  $B$  with its touching point  $\bar{\bar{x}}$ . Corresponding to  $i = 1$ ,  $\bar{\bar{x}}$  is the orthogonal projection of  $\bar{x}^1$  on  $H(\bar{\bar{x}})$ , and the straight line joining  $\bar{\bar{x}}$ ,  $\bar{\bar{x}}$  is  $L_1$ . In this 2-dimensional figure  $L_1$  is the same as  $H(\bar{\bar{x}})$ ; but in higher dimensions  $H(\bar{\bar{x}})$  will be a hyperplane and  $L_1$  will be a straight line on it.  $x^{11}, x^{12}$  are the two boundary point of  $K$  where  $L_1$  intersects  $K$ , they are the end points of  $L_1 \cap K$ . All points  $x$  on  $L_1$  satisfying the property that the descent line from it in the direction  $-c^T$  intersects  $K$  are those  $x^{01}(\alpha)$ ,  $\alpha_{11} \leq \alpha \leq \alpha_{12}$ ; i.e., those on the line segment

joining  $x^{01}(\alpha_{11}), x^{01}(\alpha_{12})$ . It can be verified that the half-line from any point on  $L_1$  outside the interval  $[x^{01}(\alpha_{11}), x^{01}(\alpha_{12})]$  in the direction  $-c^T$  does not intersect  $K$  at all. Minimizing  $f^1(\alpha)$  over  $\alpha_{11} \leq \alpha \leq \alpha_{12}$  yields the point  $x^{01}(\bar{\alpha})$  in  $K$ . My thanks to Kayse Maass for drawing this figure.

Let  $r$  be an index  $i \in T(\bar{x})$  corresponding to the maximum value for  $\|x^{0i}(\alpha_{i1}) - x^{0i}(\alpha_{i2})\|$ . Now we will minimize  $f^r(\alpha)$  over the interval  $\alpha_{r1} \leq \alpha \leq \alpha_{r2}$ . There are two approaches for carrying this out, one using a 2-variable LP formulation of this problem, and the other using a line search algorithm, which we discuss below.

**Approach 1: By solving a 2-variable LP:** The problem of minimizing  $f^r(\alpha)$  subject to  $\alpha_{r1} \leq \alpha \leq \alpha_{r2}$  is clearly the same as the 2-variable LP with variables  $\alpha, \lambda$  of minimizing  $c(x^{r1} + \alpha(x^{r2} - x^{r1}) + \lambda(-c^T)) = cx^{r1} - \lambda cc^T$ , because both  $x^{r2}, x^{r1}$  are contained on the objective plane  $H(\bar{x})$ .

$$\begin{aligned} \min \quad & c[x^{r1} + \lambda(-c^T)] \\ \text{subject to} \quad & A_i.[x^{r1} + \alpha(x^{r2} - x^{r1}) + \lambda(-c^T)] \geq b_i \quad i = 1, \dots, m \quad (4) \\ & \text{and } \lambda \geq 0, \quad \alpha_{r1} \leq \alpha \leq \alpha_{r2} \end{aligned}$$

**Approach 2: Using a Line Search algorithm:** Using the procedure described above for computing  $f^r(\alpha)$  for any given value of  $\alpha$  in the interval  $\alpha_{r1} \leq \alpha \leq \alpha_{r2}$  we can use a popular line search algorithm like ‘‘Quadratic interpolation’’ (also known as ‘‘Quadratic fit line search method’’ ) in Nonlinear Programming for solving this problem of minimizing  $f^r(\alpha)$  subject to  $\alpha_{r1} \leq \alpha \leq \alpha_{r2}$  (for example see Pages 558-560 in [1]).

Let  $\bar{\alpha}$  be the optimum value for  $\alpha$  obtained for this problem by either approach. Then the output point of this descent step is  $x^{0r}(\bar{\alpha})$ . Store the point  $x^{0r}(\bar{\alpha})$  along with its objective value  $cx^{0r}(\bar{\alpha})$  in the set  $\Gamma$ . See Figure 1.

When these line search steps are completed for all  $i \in T(\bar{x}, K)$ , this iteration is completed. The output point in this iteration is the point in the set  $\Gamma$  associated with the smallest objective value at this stage. Suppose it is  $\hat{x}$ , with objective value  $c\hat{x}$ . With  $\hat{x}$  as the initial IFS the algorithm now moves to the next iteration. The decrease in the objective value in this iteration that started with the IFS  $\hat{x}$  is the difference in the objective values at the initial IFS and the final output point in this iteration, i.e.,  $c\hat{x} - c\hat{x}$ . The algorithm is terminated in an iteration if the decrease in objective value attained in that iteration is  $\leq \epsilon$ , with the final output point in that iteration taken as the approximate optimum solution of the original LP.

### 3 Efficient Methods for Solving the 2-Variable LPs in the Centering Steps, and In Descent Steps 5.7

Here we describe methods for solving the 2-variable LPs in the centering steps (Section 2.1) and in the Descent Steps 5.7 (Section 2.2) during the algorithm for the case when  $K$  is bounded, assuming that Approach 1 is being used in both sections.

#### 3.1 Solving the 2-Variable LPs (3) Under Approach 1 in the Centering Steps in Section 2.1

The 2-variable LP to be solved is (3), with  $\gamma_{i1} < 0 < \gamma_{i2}$ , both finite and available already. Let  $\Delta_1$  denote the set of feasible solutions  $(\gamma, \delta)$  of (3) in the 2-dimensional space  $(\gamma, \delta)$  with  $\gamma$  plotted on the horizontal axis and  $\delta$  plotted on the vertical axis.

Each iteration in this method begins with an initial feasible solution on the boundary of  $\Delta_1$ . We have the initial feasible solution for (3),  $(\gamma, \delta) = (\gamma_{i1}, 0)$  as  $\delta(x_C^i(\gamma_{i1})) = 0$  since  $x_C^i(\gamma_{i1})$  is a boundary point of  $K$ . Since  $x_C^i(\gamma_{i2})$  is also a boundary point of  $K$ , we also have  $\delta(x_C^i(\gamma_{i2})) = 0$ .

The first iteration in this method begins with the initial feasible solution  $(\gamma_{i1}, 0)$  of (3) on the boundary of  $\Delta_1$ , and performs a (horizontal move + vertical move) twice and finally a diagonal move in  $\Delta_1$ ; and the current value of  $\delta$  increases in each iteration. For any given value  $\gamma$  satisfying  $\gamma_{i1} \leq \gamma \leq \gamma_{i2}$ , define  $\delta(\gamma) = \text{maximum value of } \delta \text{ such that } (\gamma, \delta) \text{ is feasible to (3)}$  = Minimum  $\{[A_i(\bar{x}(1 - \gamma) + \gamma\bar{x}) - b_i]/\|A_i\|: i = 1 \text{ to } m\}$ , and  $(\gamma, \delta(\gamma))$  is contained on the

boundary of  $\Delta_1$ .

From well known results in LP theory we know that  $\delta(\gamma)$  is piece-wise linear concave over the interval  $\gamma_{i1} \leq \gamma \leq \gamma_{i2}$ , and we have seen already that both  $\delta(\gamma_{i1}), \delta(\gamma_{i2})$  are 0.

We will now discuss a general iteration in this method beginning with the initial feasible solution  $(\gamma_0, \delta_0)$  on the boundary of  $\Delta_1$ .

**The First Horizontal Move:** Fixing  $\delta = \delta_0$ , the system of constraints in (3) becomes:

$$A_i(\bar{x}(1 - \gamma) + \gamma \hat{x}) \geq b_i + \delta_0 \|A_i\|, \quad i = 1 \text{ to } m \quad (5)$$

Find the interval of values of  $\gamma$  feasible to this system (5) using Subroutine 1. Since  $(\gamma_0, \delta_0)$  is a boundary point of  $\Delta_1$ ,  $\gamma_0$  will be one of the bounds on this interval. If  $\gamma_0$  is the unique solution of (5), then  $(\gamma_0, \delta_0)$  is an optimum solution of (5), terminate. Otherwise let  $\gamma_1$  be the midpoint of this interval; then the point  $(\gamma_1, \delta_0)$  is called **the center of**  $\Delta_1$  on the horizontal line  $\delta = \delta_0$  in the  $(\gamma, \delta)$ -space.

**The First Vertical Move:** In this move, fix  $\gamma = \gamma_1$  in (3). Then the maximum value of  $\delta$  subject to  $(\gamma_1, \delta) \in \Delta_1$  is

$$\delta_1 = \text{Minimum} \{ [A_i(\bar{x}(1 - \gamma_1) + \gamma_1 \hat{x}) - b_i] / \|A_i\| : i = 1 \text{ to } m \}$$

and the point in  $\Delta_1$  achieving this value is the boundary point  $(\gamma_1, \delta_1)$ .

**The Second (Horizontal + Vertical) Moves:** Starting with  $(\gamma_1, \delta_1)$  for (3) apply the second horizontal move as described under the first horizontal move. When  $\delta$  is fixed at  $\delta_1$ , if  $\gamma_1$  is the unique value of  $\gamma$  such that  $(\gamma, \delta_1)$  is feasible to (3); then  $(\gamma_1, \delta_1)$  is the optimum solution for (3), terminate.

Otherwise let  $\gamma_2$  be the midpoint of the interval of values of  $\gamma$  such that  $(\gamma, \delta_1)$  is feasible to (3). Then  $(\gamma_2, \delta_1)$  is called the **center of**  $\Delta_1$  on the horizontal line  $\delta = \delta_1$  in the  $(\gamma, \delta)$ -space.

Now carry out the second vertical move keeping  $\gamma$  fixed at  $\gamma_2$  to find the maximum value  $\delta_2$  of  $\delta$  in  $\Delta_1$ , as in the first vertical move. Suppose it is attained at  $(\gamma_2, \delta_2)$  on the boundary of  $\Delta_1$ .

**The Diagonal Move:** This move involves finding the maximum value of  $\delta$  for points along

the line joining the two centers of  $\Delta_1$  obtained in the two horizontal moves in this iteration,  $(\gamma_1, \delta_0), (\gamma_2, \delta_1)$ , with  $\delta_1 > \delta_0$ . The equation for the line joining  $(\gamma_1, \delta_0), (\gamma_2, \delta_1)$  in the  $(\gamma, \delta)$  space is:

$$\gamma = \gamma_1 + (\delta - \delta_0)s, \text{ where } s = (\gamma_2 - \gamma_1)/(\delta_1 - \delta_0).$$

Substituting this expression for  $\gamma$  in (3) leads to the following system of linear inequalities in  $\delta$ .

$$\delta[||A_i|| - sA_i(\bar{x} - \hat{x})] \leq A_i\bar{x} - b_i + (\gamma_1 - s\delta_0)A_i(\bar{x} - \hat{x}), \quad i = 1, \dots, m.$$

Find the maximum value of  $\delta$ ,  $\delta_3$  feasible to this system of inequalities in  $\delta$  using Subroutine 1. So,  $\delta_3$  is the maximum value of  $\delta$  in  $\Delta_1$  on the straight line joining  $(\gamma_1, \delta_0), (\gamma_2, \delta_1)$ . Let  $\gamma_3 = \gamma_1 + (\delta_3 - \delta_0)s$ ,  $(\gamma_3, \delta_3)$  is the point of intersection of this straight line with  $\Delta_1$ .

Let  $\delta_4 = \text{maximum } \{\delta_2, \delta_3\}$ , and let  $(\gamma_4, \delta_4)$  be the point among  $(\gamma_2, \delta_2), (\gamma_3, \delta_3)$  associated with it, i.e.,  $\gamma_4 = \gamma_2$  if  $\delta_4 = \delta_2$ , or  $\gamma_4 = \gamma_3$  if  $\delta_4 = \delta_3$ . Then  $(\gamma_4, \delta_4)$  is the output of this iteration in this method, with this point go to the next iteration.

When the improvement in the value of  $\delta$  becomes small in an iteration, terminate the method with the output in that iteration as an approximate optimum for (3).

### 3.2 Solving the 2-Variable LPs (4) Under Approach 1 in the Descent Steps D5.7 in Section 2.2

We will solve the equivalent problem:

$$\begin{aligned} &\text{Maximize} && \lambda && \text{subject to} \\ &\lambda(-A_i c^T) + \alpha(A_i(x^{r2} - x^{r1})) \geq b_i - A_i x^{r1}, && i = 1, \dots, m && (6) \\ &\lambda \geq 0, && \alpha_{r1} \leq \alpha \leq \alpha_{r2} \end{aligned}$$

where  $\alpha_{r1} \leq 0$ ,  $\alpha_{r2} \geq 1$  are already computed in D5.7. When a descent step is taken from the point  $x^{r1} + \alpha(x^{r2} - x^{r1})$  in the direction  $-c^T$  for some  $\alpha_{r1} \leq \alpha \leq \alpha_{r2}$ , the maximum step length possible while keeping the output point inside  $K$  is:

$$\lambda(\alpha) = \text{minimum}\{(b_p - A_p(x^{r1} + \alpha(x^{r2} - x^{r1}))/A_p(-c^T): \text{ over } 1 \leq p \leq m \text{ satisfying } A_p(-c^T) < 0\}.$$

Find  $\lambda(\alpha_{r1})$ ,  $\lambda(\alpha_{r2})$ . and  $\lambda_s = \text{maximum}\{\lambda(\alpha_{r1}), \lambda(\alpha_{r2})\}$ .

We will now describe the application of the method described in Section 3.1, to solve the 2-variable LP (6) in the  $(\lambda, \alpha)$  space, with  $\alpha$  plotted along the horizontal axis, and  $\lambda$  on the vertical axis. The first iteration begins with the line  $\lambda = \text{maximum}\{\lambda(\alpha_{r1}), \lambda(\alpha_{r2})\}$ .

We will now describe the general iteration in this method beginning on the line  $\lambda = \lambda_0$ .

**The First Horizontal Move:** Fix  $\lambda = \lambda_0$  in (6), and find the interval of values of  $\alpha$  feasible to it using Subroutine 1. If  $\alpha = \alpha_1$  is the unique solution of this system, then  $(\lambda_0, \alpha_1)$  is the optimum solution of (6), terminate. Otherwise, let  $\alpha_1$  be the midpoint of this interval. Then  $(\lambda_0, \alpha_1)$  is called the **center** of the set of feasible solutions of (6) on the horizontal line  $\lambda = \lambda_0$  in the  $(\lambda, \alpha)$ -space.

**The First Vertical Move:** Compute  $\lambda(\alpha_1) = \lambda_1$  say.

**The Second Horizontal Move:** Fix  $\lambda = \lambda_1$  in (6), and find the interval of values of  $\alpha$  feasible to it using Subroutine 1. If this solution is  $\alpha_2$  which is unique, then  $(\lambda_1, \alpha_2)$  is an optimum solution of (6), terminate. Otherwise, let  $\alpha_2$  be the midpoint of this interval. Then  $(\lambda_1, \alpha_2)$  is called the **center** of the set of feasible solutions of (6) on the horizontal line  $\lambda = \lambda_1$  in the  $(\lambda, \alpha)$ -space.

**The Second Vertical Move:** Compute  $\lambda(\alpha_2) = \lambda_2$  say. The point corresponding to it feasible to (6) is  $(\lambda_2, \alpha_2)$ .

**The Diagonal Move:** This move finds the maximum value of  $\lambda$  in (6) for points along the straight line joining the two centers  $(\lambda_0, \alpha_1)$ ,  $(\lambda_1, \alpha_2)$ . The equation describing this straight line is:

$$\lambda = \lambda_0 + (\alpha - \alpha_1)((\lambda_1 - \lambda_0)/(\alpha_2 - \alpha_1)).$$

Substitute this expression for  $\lambda$  in (6), and find the maximum value of  $\alpha$  feasible to the resulting system,  $\alpha_3$  say, using Subroutine 1. Let  $\lambda_3 = \lambda_0 + (\alpha_3 - \alpha_1)((\lambda_1 - \lambda_0)/(\alpha_2 - \alpha_1))$ .

If  $\lambda_3 > \lambda_2$ , then  $(\lambda_3, \alpha_3)$  is the output point in this iteration. On the other hand if  $\lambda_2 > \lambda_3$ , then  $(\lambda_2, \alpha_2)$  is the output point in this iteration. Go to the next iteration beginning with the horizontal line through the output point.

Terminate the method with the output point in an iteration as an approximate optimum

for (6) when the improvement in the value of  $\lambda$  in the iteration becomes small. If  $(\bar{\lambda}, \bar{\alpha})$  is the output point in this method, then  $x^{r1} + \bar{\alpha}(x^{r2} - x^{r1}) + (\bar{\lambda} - \epsilon)(-c^T)$  is the output point of this method for (4) in the space of the original variables  $x$ .

## 4 Computational Results

## 5 References

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