SOME NP-COMPLETE PROBLEMS IN LINEAR PROGRAMMING

R. CHANDRASEKARAN, Santosh N. KABADI and Katta G. MURTY *

School of Management and Administration, University of Texas at Dallas, P.O. Box 688, Richardson, TX 75080, U.S.A.

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Degeneracy checking in linear programming is NP-complete. So is the problem of checking whether there exists a basic feasible solution with a specified objective value.

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1. Degeneracy testing

Consider the general linear program in standard form, with integer data

\[
\begin{align*}
\text{minimize} & \quad Z(x) = cx, \\
\text{subject to} & \quad Ax = b, \\
& \quad x \geq 0
\end{align*}
\]

where $A$ is a matrix of order $m \times n$ and rank $m$. This problem is said to be degenerate, if there exists a basis $B$ for (1) satisfying the property that at least one component in the vector $B^{-1}b$ is zero. See [1-10,13]. Degeneracy in linear programming was studied extensively, because of the problem of cycling that it can introduce in the simplex algorithm, thereby preventing the simplex algorithm from terminating in a finite number of steps unless special measures are taken to resolve degeneracy [1-10,13]. If (1) is degenerate, the point $b$ must be in a subspace of $\mathbb{R}^m$ spanned by some subset of $(m-1)$ column vectors of $A$. Therefore, when $A$, $b$, $c$ are allowed to be real or rational, in a statistical sense, (1) will be nondegenerate almost always. Also, even if (1) is degenerate, when $b$ is modified to $b'(e) = b + (e, e^2, \ldots, e^m)^T$, there exists an $e_i > 0$ such that whenever $0 < e < e_i$, the modified problem is nondegenerate. Thus, a minor perturbation will make (1) nondegenerate, and methods for resolving degeneracy in the simplex and other pivotal algorithms have been developed based on such perturbations [2,3,7,10,13]. In spite of all these statistical arguments, it has been observed that most linear programming models constructed in practical applications tend to be degenerate [3].

Here we study the computational complexity of checking whether a given instance of (1) is degenerate. We will first discuss a combinatorial optimization problem. Let $a = \{a_1, \ldots, a_m\}$, $b = \{b_1, \ldots, b_n\}$ be two given finite sets of positive integers. The term \textit{equal partial sums} denotes the combinatorial optimization problem: given the sets $a$, $b$, find whether there exist subsets $I$, $J$ satisfying $\emptyset \neq I \subseteq \{1, \ldots, m\}$, $\emptyset \neq J \subseteq \{1, \ldots, n\}$, such that $\sum_{i \in I} a_i = \sum_{j \in J} b_j$.

\textbf{Lemma.} The problem \textit{equal partial sums} is NP-complete.

\textbf{Proof.} Clearly, the problem \textit{equal partial sums} is in NP. Consider the subset sum problem: given positive integers $d_1, \ldots, d_p, d_0$, check whether there exists a subset $I \subseteq \{1, \ldots, p\}$ satisfying $\sum_{i \in I} d_i = d_0$. Here, is $\sum_{i=1}^p d_i = d_0$, $I = \{1, \ldots, p\}$ provides an answer to the subset sum problem in the affirmative, so without any loss of generality we can assume that $\sum_{i=1}^p d_i > d_0$. Let $a = 1 + \sum_{i=1}^p d_i$. In this case, the subset sum problem is equivalent to the \textit{equal partial sums} problem with $a = \{d_1, \ldots, d_p\}$, $b = \{d_0, a\}$. Thus, the subset sum problem is a special case of the \textit{equal partial sums} problem. Since the subset sum problem is NP-complete [11,12], these facts imply that so is the \textit{equal partial sums} problem.
Theorem. Degeneracy testing is NP-complete.

Proof. If a basis $B$ for (1) exists exhibiting degeneracy, a nondeterministic algorithm can select this basis one column at a time in at most $m$ steps. Thus, degeneracy testing is in NP.

Consider the special case of (1), known as the transportation problem, in which the constraints are of the form

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1 \text{ to } m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1 \text{ to } n,$$

$$x_{ij} \geq 0 \quad \text{for all } i, j \tag{2}$$

where $a_1, \ldots, a_m; b_1, \ldots, b_n$ are given positive integers satisfying $\Sigma_{i=1}^{m} a_i = \Sigma_{j=1}^{n} b_j$. It is known [8,9,13] that (2) is degenerate iff there exists proper subsets $\emptyset \neq I \subset \{1, \ldots, m\}$, $\emptyset \neq J \subset \{1, \ldots, n\}$ satisfying $\Sigma_{i \in I} a_i = \Sigma_{j \in J} b_j$. Thus checking whether (2) is degenerate is equivalent to the equal partial sums problem with $a = \{a_1, \ldots, a_m\}$, $b = \{b_1, \ldots, b_n\}$. By lemma, these facts clearly imply that degeneracy testing is NP-complete.

A degenerate feasible basis for (1) is a basis $B$ for (1) satisfying $B^{-1}b \geq 0$, and at least one component of $B^{-1}b$ is zero. It is possible for (1) to be degenerate, and yet there may not exist a degenerate feasible basis for (1). For the special case of the transportation problem (2), it can be shown that a degenerate feasible basis exists iff the equal partial sums problem with $a = \{a_1, \ldots, a_m\}$, $b = \{b_1, \ldots, b_n\}$ has a solution. This leads to the following:

Corollary 1. The problem of checking whether there exists a degenerate feasible basis for (1), is NP-complete.

Corollary 2. Degeneracy testing is NP-complete even for the special case of the transportation problem.

2. Extreme point with a specified objective value

Given the LP (1) with integer data, and a rational number $\theta$ expressed as a ratio in smallest terms, this problem is to check whether there exists a basic feasible solution of (1) at which the objective function assumes the value of $\theta$. Clearly this problem is in NP and it can be shown to be NP-complete by showing the problem of testing for a degenerate feasible basis in (1) to be a special case of it. We now show that extreme point with a specified objective value problem is NP-complete even for the special case of Assignment problem:

Consider the subset sum problem with data $d_1, \ldots, d_p; d_0$ discussed above. Let $C = (c_{ij})$ be a $2p \times 2p$ matrix:

$$C = \begin{bmatrix} d_1 & d_2 & \ldots & d_p \\ d_1 & d_2 & \ldots & d_p \\ \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & \ldots & d_p \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$ 

The last $p$ columns of $C$ are zero. The first $p$ columns of $C$ are all equal to $(d_1, d_2, \ldots, d_p, 0, 0, \ldots, 0)^T \in \mathbb{R}^{2p}$. Clearly, the answer to the subset sum problem is in the affirmative iff there exists an assignment of order $2p$ for which the objective value, with $C$ as the cost matrix is $d_0$. Since the assignments are the extreme points associated with the assignment problem, this shows that the subset sum problem is a special case of the extreme point with a specified objective value problem. So the extreme point with a specified objective value problem is NP-complete, even when restricted to the assignment problem.

3. Singular principal submatrix problem

Given a square, nonsingular, integer matrix, $A$, consider the problem of checking whether there exists a singular principal submatrix of $A$. This problem is clearly in NP. To show that it is NP-complete, consider again the subset sum problem with data $d_1, d_2, \ldots, d_p; d_0$, as discussed before. If $d_0 \geq \Sigma_{i=1}^{p} d_i$, the problem becomes trivial. So, without loss of generality, let $d_0 < \Sigma_{i=1}^{p} d_i$.

Now let us define a square, non-singular matrix, $A$, as follows:

$$A = \begin{bmatrix} d_0 & d_1 & d_2 & \ldots & d_p \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$ 

Clearly, the answer to the subset sum problem is affirmative iff there exists a principal submatrix of $A$, which is singular. This leads to the following:
Corollary: Given two square, nonsingular, integer matrices $A, B$ of order $n$, let $A_j, B_j$ denote the $j$th column vectors of $A, B$ resp. Then, checking whether there exists a set of columns $\{D_1, ..., D_n\}$, which is linearly dependent, with $D_j \in \{A_j, B_j\}$, is NP-complete.

4. Bilinear problem

The problem, considered, is

minimize $y^T D z + p^T y + q^T z$,
subject to $B y = d$,
$E z = 0$,
y, $z \geq 0$.

(3)

We show below that degeneracy testing in (1) can be posed as a bilinear problem of the type (3).

Define 0–1 variables

$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \text{ in a solution for (1)}, \\ 0 & \text{if } x_j = 0. \end{cases}$$

Then the standard trick of transforming (1), using 0–1 variables $y_j$ to count the number of positive variables, $x_j$, in a solution in (1) is well known. This leads to a system, say (4) of linear constraints in $x$, $y$, including $0 \leq y_j \leq 1$ for all $j$. To make sure that all $y_j$ are either 0 or 1, make the objective function equal to $\alpha \sum_{j=1}^{n} y_j (1-y_j)$, where $\alpha$ is a suitably large positive penalty parameter.

Now, put two sets of system (4) together. In one call the variables as $x'$, $y$. In the other, call them $x''$, $z$. Call this combined system as (5).

The question: Does (1) have a feasible solution with number of positive $x_j < m - 1$ is equivalent to the following:

Does (5) have a feasible solution in which all $y_j$, $z_j$ are integer and $y = z$?

This problem is the same as that of minimizing

$$\sum_{j=1}^{n} y_j + \alpha \sum_{j=1}^{n} y_j (1-y_j)$$

$$+ \alpha \sum_{j=1}^{n} z_j (1-z_j) + \alpha \sum_{j=1}^{n} (y_j - z_j)^2$$

(6)

subject to (5).

On simplification, (6) becomes

$$-2 \alpha \sum_{j=1}^{n} y_j z_j + \alpha \left( \sum_{j=1}^{n} (y_j + z_j) \right) + \sum_{j=1}^{n} y_j.$$  

(7)

Which is clearly a bilinear objective function. So, the bilinear problem (1) is NP-hard.

5. Some open problems

Here is a problem related to the Hirsch Conjecture whose status is unknown. Given a convex polyhedron specified by linear inequalities with integer data $Ax \geq b$

and two extreme points $x^1, x^2$ on it and a positive integer $\alpha$, the problem is to check whether there exists an edge path in the polyhedron between $x^1, x^2$, containing $\alpha$ or less edges. The computational complexity of this problem is still unknown.

A second problem of interest is the following: Suppose we are given an integer matrix $A$ of order $m \times n$. The problem of finding a maximum cardinality linearly independent subset of column vectors of $A$ can of course be solved efficiently, using pivot step in at most $O(n^3)$ time. The complementary problem of finding a minimum cardinality linearly dependent subset of column vectors of $A$ seems to be hard in general. A specific problem of interest is, given that rank of $A$ is $m$, checking whether there exists a subset of $m$ columns of $A$, which is linearly dependent. The problem is simple when $A$ is unimodular. But its computational complexity is not known in general.

References


