Chapter 5

THE PARAMETRIC LINEAR COMPLEMENTARITY PROBLEM

Let M be a given square matrix of order n and let b, b^* be given column vectors in $\mathbf{R}^n$. Let $q(\lambda) = b + \lambda b^*$. Assuming that $b^* \neq 0$, $q(\lambda)$ traces a straight line in $\mathbf{R}^n$, $\mathbf{L} = \{x : x = q(\lambda), \text{ for some } \lambda\}$, as λ takes all real values. We consider the following parametric LCP: find w, z satisfying

$$\begin{aligned} w - Mz &= q(\lambda) = b + \lambda b^* \\ w \geqq 0&, \ z \geqq 0 \\ w^T z &= 0 \end{aligned} \tag{5.1}$$

as functions of λ, for each value of λ in some specified interval. Here we discuss an algorithm developed in [5.12] by K. G. Murty for obtaining a solution of this parametric LCP as a function of λ. This algorithm is most useful when M is a P-matrix. This algorithm solves the LCP $(q(\lambda), M)$ for some fixed value of λ by any method (such as the complementary pivot method, or the principal pivoting methods), and then obtains solutions for the parametric LCP for all values of λ using only a series of single principal pivot steps.

The Algorithm

Step 1: Choose a value λ_0, and fix λ at λ_0 (λ_0 could be equal to zero), and solve the LCP $(q(\lambda_0), M)$ by any one of the algorithms discussed earlier, and obtain a complementary feasible basic vector for it. With this complementary feasible basic vector for (5.1) when $\lambda = \lambda_0$, go to Step 2.

Step 2: Determine the range of values of λ for which the present complementary basic vector remains feasible. The procedure for doing this is the same as in parametric right hand side LP, and it is as follows: Let $(y_1, \dots, y_n)$, where $y_j \in \{w_j, z_j\}$ for each $j = 1$ to n, be the present complementary basic vector. Let $\beta = (\beta_{ij})$ be the inverse of the present complementary basis. Let $\overline{b}$, $\overline{b}^*$ be the present updated right hand side constants vectors, that is $\overline{b} = \beta b$, $\overline{b}^* = \beta b^*$. Compute $\underline{\lambda}$, $\overline{\lambda}$; the **lower and upper characteristic values associated with the present complementary basic vector**, from the following.

$$\begin{aligned}
\underline{\lambda} &= -\infty, \text{ if } \overline{b}_i^* \leqq 0 \text{ for all } i \\
&= \text{Maximum } \{-\overline{b}_i/\overline{b}_i^* : i \text{ such that } \overline{b}_i^* > 0\}, \text{ otherwise} \\
\overline{\lambda} &= +\infty, \text{ if } \overline{b}_i^* \geqq 0 \text{ for all } i \\
&= \text{Minimum } \{-\overline{b}_i/\overline{b}_i^* : i \text{ such that } \overline{b}_i^* < 0\}, \text{ otherwise}.
\end{aligned} \tag{5.2}$$

Since the present complementary basic vector is feasible for (5.1) for at least one value of λ, we will have $\underline{\lambda} \leqq \overline{\lambda}$, and for all values of λ in the closed interval $\underline{\lambda} \leqq \lambda \leqq \overline{\lambda}$, the present complementary basic vector remains feasible, and hence the solution

$$\begin{gathered}
\text{Present } i\text{th basic variable } y_i = \overline{b}_i + \lambda \overline{b}_i^*, i = 1 \text{ to } n \\
\text{Complement of } y_i, t_i = 0, i = 1 \text{ to } n
\end{gathered} \tag{5.3}$$

is a solution of the parametric LCP $(q(\lambda), M)$. Go to Step 3 or 4 if it is required to find the solutions of the parametric LCP $(q(\lambda), M)$ for values of $\lambda > \overline{\lambda}$, or for values of $\lambda < \underline{\lambda}$ respectively.

Step 3: We come to this step when we have a complementary basic vector, $y = (y_1, \dots, y_n)$ say, for which the upper characteristic value is $\overline{\lambda}$, and it is required to find solutions of the parametric LCP $(q(\lambda), M)$, for values of $\lambda > \overline{\lambda}$. Let $\overline{b}$, $\overline{b}^*$ be the present updated right hand side constant vectors. Find out $\mathbf{J} = \{i : i$ ties for the minimum in (5.2) for determining $\overline{\lambda}\}$, $r =$ maximum $\{i : i \in \mathbf{J}\}$. So $\overline{b}_r^* < 0$ and $-\overline{b}_r/\overline{b}_r^* = \overline{\lambda}$. The value of the rth basic variable y_r, in the solution in (5.3) is zero when $\lambda = \overline{\lambda}$, and it becomes negative when $\lambda > \overline{\lambda}$. Let t_r be the complement of y_r and let $\overline{A}_{.r} = (\overline{a}_{1r}, \dots, \overline{a}_{nr})^T$ be its updated column vector. If $\overline{a}_{rr} < 0$, perform a single principal pivot step in position r in y leading to the complementary basic vector $u = (y_1, \dots, y_{r-1}, t_r, y_{r+1}, \dots, y_n)$. Both y and u have the same BFS when $\lambda = \overline{\lambda}$ (since $y_r = 0$ when $\lambda = \overline{\lambda}$ in the solution in (5.3)). u is a complementary feasible basic vector for (5.1) when $\lambda = \overline{\lambda}$. The value of t_r in the basic solution of (5.1) with respect to u is $(\overline{b}_r/\overline{a}_{rr}) + \lambda(\overline{b}^*/\overline{a}_{rr})$, this quantity is 0 when $\lambda = \overline{\lambda}$, and since $\overline{a}_{rr} < 0$, $\overline{b}_r^* < 0$, we verify that this quantity is positive when $\lambda > \overline{\lambda}$. From this it can be verified that the lower characteristic value for u is $\overline{\lambda} =$ upper characteristic value for y. With u, go back to Step 2.

If $\overline{a}_{rr} \geqq 0$, either the single principal pivot step in position r cannot be carried out (when $\overline{a}_{rr} = 0$); or even after it is carried out, the new rth basic variable continues to be negative when $\lambda > \overline{\lambda}$ in the new basic solution (which happens when $\overline{a}_{rr} > 0$). Thus in this case, the algorithm is unable to solve the parametric LCP $(q, (\lambda), M)$ for $\lambda > \overline{\lambda}$, it is even unable to determine whether there exists a solution to the LCP $(q,(\lambda),M)$ or not when $\lambda > \overline{\lambda}$.

Step 4: We come to this step when we have a complementary basic vector, $y = (y_1,\dots,y_n)$ say, for which the lower characteristic value is $\underline{\lambda}$, and it is required to find solutions of the parametric LCP $(q,(\lambda),M)$ for values of $\lambda < \underline{\lambda}$. Let $\mathbf{J} = \{i : i$ ties for the maximum in (5.2) for determining $\underline{\lambda}\}$, $r =$ maximum $\{i : i \in \mathbf{J}\}$. Let t_r be the complement of y_r and let $\overline{A}_{.r} = (\overline{a}_{1r},\dots,\overline{a}_{nr})^T$ be its updated column vector. If $\overline{a}_{rr} < 0$, perform a single principal pivot step in position r in y. This leads to the next complementary feasible basic vector for which $\underline{\lambda}$ is the upper characteristic value, continue with it in the same way. If $\overline{a}_{rr} \geqq 0$, this algorithm is unable to solve, or even determine whether a solution exists for the parametric LCP $(q,(\lambda),M)$ when $\lambda < \underline{\lambda}$.

Example 5.1

Consider the parametric LCP $(q,(\lambda) = b + \lambda b^*, M)$, for which the original tableau is

w_1	w_2	w_3	z_1	z_2	z_3	b	b^*
1	0	0	−1	0	0	8	−1
0	1	0	−2	−1	0	4	−1
0	0	1	−2	−2	−1	2	−1

When $\lambda = 0$, (w_1,w_2,w_3) is a complementary feasible basic vector for this problem. The inverse tableau corresponding to this is:

First Inverse Tableau

Basic Variable	Inverse of the Complementary Basis			$\overline{b}$	$\overline{b}^*$	$-\overline{b}_i/\overline{b}^*_i$ for i such that $\overline{b}^*_i < 0$	$-\overline{b}_i/\overline{b}^*_i$ for i such that $\overline{b}^*_i > 0$	Range of Feasibility	Pivot Column z_3
w_1	1	0	0	8	−1	8		$-\infty < \lambda \leqq 2$	0
w_2	0	1	0	4	−1	4			0
w_3	0	0	1	2	−1	2			**[−1]**

So in the range $-\infty < \lambda \leqq 2$, $(w = (8-\lambda, 4-\lambda, 2-\lambda)^T, z = 0)$ is a solution of this parametric LCP. To find out solutions of this parametric LCP when $\lambda > 2$, we have to make a single principal pivot step in position 3.

The updated column vector of z_3 is $\overline{A}_{.3} = (0,0,-1)$. $\overline{a}_{33} = -1$, and hence we can continue. The pivot column is already entered by the side of the first inverse tableau. Performing the pivot leads to the next inverse tableau.

Second Inverse Tableau

Basic Variable	Inverse of the Complementary Basis			$\overline{b}$	$\overline{b}^*$	$-\overline{b}_i/\overline{b}_i^*$ for i such that $\overline{b}_i^* < 0$	$\overline{b}_i^* > 0$	Range of Feasibility
w_1	1	0	0	8	−1	8		$2 < \lambda \leqq 4$
w_2	0	1	0	4	−1	4		
z_3	0	0	−1	−2	1		2	
						$\overline{\lambda} = 4$	$= \underline{\lambda} = 2$	

So in the range $2 \leqq \lambda \leqq 4$, the solution $(w_1, w_2, z_3) = (8 - \lambda, 4 - \lambda, -2 + \lambda)$, $(z_1, z_2, w_3) = (0, 0, 0)$ is a solution of this parametric LCP $(q(\lambda), M)$. Continuing in the same way, we get the following solutions for this problem summarized in the table below.

Optimality Range	Complementary Feasible Basic Vector	Complementary Solution $(w^T; z^T)$
$-\infty < \lambda < 2$	(w_1, w_2, w_3)	$(8 - \lambda, 4 - \lambda, 2 - \lambda; 0, 0, 0)$
$2 \leqq \lambda \leqq 4$	(w_1, w_2, z_3)	$(8 - \lambda, 4 - \lambda, 0; 0, 0, -2 + \lambda)$
$4 \leqq \lambda \leqq 6$	(w_1, z_2, z_3)	$(8 - \lambda, 0, 0; 0, -4 + \lambda, 6 - \lambda)$
$6 \leqq \lambda \leqq 8$	(w_1, z_2, w_3)	$(8 - \lambda, 0, -6 + \lambda; 0, -4 + \lambda, 0)$
$8 \leqq \lambda \leqq 10$	(z_1, z_2, w_3)	$(0, 0, -6 + \lambda; -8 + \lambda, -4 + \lambda, 0)$
$10 \leqq \lambda \leqq 12$	(z_1, z_2, z_3)	$(0, 0, 0; -8 + \lambda, 12 - \lambda, -10 + \lambda)$
$12 \leqq \lambda \leqq 14$	(z_1, w_2, z_3)	$(0, -12 + \lambda, 0; -8 + \lambda, 0, 14 - \lambda)$
$14 \leqq \lambda$	(z_1, w_2, w_3)	$(0, -12 + \lambda, -14 + \lambda; -8 + \lambda, 0, 0)$

Example 5.2

Consider the parametric LCP $(q(\lambda) = b + \lambda b^*, M)$ for which the original tableau is:

w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	b	b^*
1	0	0	0	−1	1	1	1	3	−2
0	1	0	0	1	−1	1	1	5	−4
0	0	1	0	−1	−1	−2	0	−9	5
0	0	0	1	−1	−1	0	−2	−5	3

Putting $\lambda = 0$, we verify that this LCP is the same as the one solved in Example 2.8. The complementary feasible basic vector obtained for this problem (when $\lambda = 0$) in Example 2.8 is (z_1, z_2, z_3, z_4). The inverse tableau corresponding to (z_1, z_2, z_3, z_4) is

Basic Variable	Inverse of the Complementary Basis				$\bar{b}$	$\bar{b}^*$	$-\bar{b}_i/\bar{b}_i^*$ for i such that $\bar{b}_i^* < 0$	$-\bar{b}_i/\bar{b}_i^*$ for i such that $\bar{b}_i^* > 0$	Range of Feasibility
z_1	−1/2	0	−1/4	−1/4	2	−1	2		
z_2	0	−1/2	−1/4	−1/4	1	0			$\lambda \leqq 1$
z_3	1/4	1/4	−1/4	1/4	3	−2	3/2		
z_4	1/4	1/4	1/4	1/4	1	−1	1		
							Minimum $= \bar{\lambda} = 1$	Maximum $= \underline{\lambda} = -\infty$	

So when $\lambda \leqq 1$, the solution $(w = (w_1, w_2, w_3, w_4) = 0,\ z = (z_1, z_2, z_3, z_4) = (2 - \lambda, 1, 3 - 2\lambda, 1 - \lambda))$ is a solution of this parametric LCP $(q(\lambda), M)$. To look for solutions when $\lambda > 1$, we have to make a single principal pivot step in position 4. The updated column vector of w_4 is $\bar{A}_{.4} = (-1/4, -1/4, 1/4, 1/4)$. So $\bar{a}_{44} = 1/4 > 0$. Since $\bar{a}_{44}$ is strictly positive, the algorithm discussed above is unable to process this parametric LCP $(q(\lambda), M)$ when $\lambda > 1$.

Theorem 5.1 *Let M be a given P-matrix of order n. Consider the parametric LCP $(q(\lambda), M)$. The algorithm discussed above finds solutions of this parametric LCP for all real values of λ in a finite number of pivot steps. Also, for each λ, the solution obtained is the unique solution of this parametric LCP for that value of λ.*

Proof. In the notation of the algorithm, let $y = (y_1, \ldots, y_n)$ be the complementary basic vector in Step 2 at some stage of the algorithm, for which the range of feasibility is $\underline{\lambda} \leqq \lambda \leqq \overline{\lambda}$. In order to find out solutions for $\lambda > \overline{\lambda}$, suppose we have to make a single principal pivot step in position r. Let t_r be the complement of y_r and let $\overline{A}_{.r}$ be the updated column vector of t_r. By Corollary 3.5 the method can continue. Let the BFS with respect to y be the one given in (5.3). Let $\mathbf{T} = \{i : i \text{ such that } \overline{b}_i + \overline{\lambda}\overline{b}^*_i = 0\}$. Clearly $r \in \mathbf{T}$ and in fact the set $\mathbf{J}$ defined in Step 3 at this stage satisfies $\mathbf{J} \subset \mathbf{T}$. As long as λ remains fixed at $\overline{\lambda}$, any principal pivot steps performed on positions in $\mathbf{T}$ will not change the basic solution (because when the basic variable in the pivot row is 0 in the basic solution, the pivot step is a degenerate pivot step that leaves the basic solution unchanged). Let $u = (u_1, \dots, u_n)$ be any complementary basic vector satisfying the property that $u_i = y_i$ for $i \notin \mathbf{T}$, $u_i = y_i$ or its complement for $i \in \mathbf{T}$. Suppose the updated right hand side constant vectors with respect to u are $\hat{b}$, $\hat{b}^*$. By the above argument, the basic solution of (5.1) with respect to u at $\lambda = \overline{\lambda}$ is

$$u_i = \hat{b}_i + \overline{\lambda}\hat{b}^*_i = \overline{b}_i + \overline{\lambda}\,\overline{b}^*_i, \;\; i = 1 \text{ to } n$$
$$(\text{Complement of } u_i) = 0, \;\; i = 1 \text{ to } n.$$

So $\hat{b}_i + \overline{\lambda}\hat{b}_i = 0$ for $i \in \mathbf{T}$ and > 0 for $i \notin \mathbf{T}$. So the upper characteristic point associated with u is $> \overline{\lambda}$ iff $\hat{b}^*_i \geqq 0$ for all $i \in \mathbf{T}$. Thus, if $\mathbf{T}$ is a singleton set, the pivot step carried out in Step 3 at this stage is guaranteed to produce a complementary feasible basic vector for which the upper characteristic value is $> \overline{\lambda}$. If $\mathbf{T}$ has 2 or more elements, let $\omega = (w_i, i \in \mathbf{T})$, $\xi = (z_i, i \in \mathbf{T})$, $\mathcal{M}$ the principal submatrix of M corresponding to the subset $\mathbf{T}$, and $\gamma = (\overline{b}^*_i, i \in \mathbf{T})$. Consider the LCP $(\gamma, \mathcal{M})$ in the variables (ω, ξ). Since $\mathcal{M}$ is a P-matrix, by Theorem 4.1, the LCP $(\gamma, \mathcal{M})$ can be solved by Principal Pivoting Method I in a finite number of pivot steps without cycling, starting with the complementary basic vector $(y_i, i \in \mathbf{T})$ until a complementary basic vector is obtained for it, with respect to which the updated γ is $\geqq 0$. The choice of the pivot row r in Step 3 of the parametric algorithm implies that when it is continued from the canonical tableau of (5.1) with respect to y, keeping $\lambda = \overline{\lambda}$, it will go through exactly the same sequence of pivotal exchanges as in the LCP $(\gamma, \mathcal{M})$, when it is solved by Principal Pivoting Method I, until we obtain a complementary feasible basic vector, $u = (u_1, \dots, u_n)$ say, satisfying the property that the updated b^*_i with respect to u is $\geqq 0$ for each $i \in \mathbf{T}$. By the above argument the upper characteristic value of u is $> \overline{\lambda}$, and hence when we reach the basic vector u, we are able to strictly increase the value of λ beyond $\overline{\lambda}$. Also, once we cross the interval of feasibility of a complementary basic vector in this parametric algorithm, we will never encounter this basic vector again. We can apply the same argument in Step 4 for decreasing λ below $\underline{\lambda}$. Continuing in this way, since there are only $2^{\mathbf{n}}$ complementary basic vectors, these arguments imply that after at most a finite number (less than $2^{\mathbf{n}}$) of pivot steps, we will obtain solutions of the parametric LCP $(q(\lambda), M)$ for all λ.

The fact that the solution obtained is the unique solution for each λ, follows from Theorem 3.13.

□

When there are ties for the i that attains the minimum in (5.2) of Step 3 and the pivot row is chosen among $i \in \mathbf{J}$ arbitrarily (instead of choosing it as the bottommost as mentioned in Step 3), cycling can occur at this value of $\lambda = \overline{\lambda}$, as shown in the following example due to A. Gana [5.6]. He considers the parametric LCP with the following data

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad b^* = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

Starting with the complementary feasible basic vector (w_1, w_2, w_3) when $\lambda = 0$, we want to solve this problem for all $\lambda \geqq 0$. Here is a sequence of complementary basic vectors obtained when the pivot row in Step 3 is chosen among $i \in \mathbf{J}$ arbitrarily. Pivot elements are in a box.

Basic Variables	w_1	w_2	w_3	z_1	z_2	z_3	b	b^*	Feasibility Interval
w_1	1	0	0	$\boxed{-1}$	−2	0	1	−1	
w_2	0	1	0	0	−1	−2	1	−1	$0 \leqq \lambda \leqq 1$
w_3	0	0	1	−2	0	−1	1	−1	
z_1	−1	0	0	1	2	0	−1	1	
w_2	0	1	0	0	$\boxed{-1}$	−2	1	−1	$1 \leqq \lambda \leqq 1$
w_3	−2	0	1	0	4	−1	−1	1	
z_1	$\boxed{-1}$	2	0	1	0	−4	1	−1	
z_2	0	−1	0	0	1	2	−1	1	$1 \leqq \lambda \leqq 1$
w_3	−2	4	1	0	0	−9	3	−3	
w_1	1	−2	0	−1	0	4	−1	1	
z_2	0	−1	0	0	1	2	−1	1	$1 \leqq \lambda \leqq 1$
w_3	0	0	1	−2	0	$\boxed{-1}$	1	−1	
w_1	1	−2	4	−9	0	0	3	−3	
z_2	0	$\boxed{-1}$	2	−4	1	0	1	−1	$1 \leqq \lambda \leqq 1$
z_3	0	0	−1	2	0	1	−1	1	
w_1	1	0	0	$\boxed{-1}$	−2	0	1	−1	
w_2	0	1	−2	4	−1	0	−1	1	$1 \leqq \lambda \leqq 1$
z_3	0	0	−1	2	0	1	−1	1	

Basic Variables	w_1	w_2	w_3	z_1	z_2	z_3	b	b^*	Feasibility Interval
z_1	-1	0	0	1	2	0	-1	1	
w_2	4	1	-2	0	-9	0	3	-3	$1 \leqq \lambda \leqq 1$
z_3	2	0	$\boxed{-1}$	0	4	1	1	-1	
z_1	-1	0	0	1	2	0	-1	1	
w_2	0	1	0	0	-1	-2	1	-1	$1 \leqq \lambda \leqq 1$
w_3	-2	0	1	0	4	-1	-1	1	

The complementary basic vector (z_1, w_2, w_3) repeated at $\lambda = 1$, and hence cycling has occurred, and the execution can go through this cycle repeatedly without ever being able to increase λ beyond 1. Theorem 5.1 indicates that if the pivot row is chosen as mentioned in Steps 3, 4 of the parametric algorithm, this cycling cannot occur.

Geometric Interpretation

Let M be a given square matrix of order n. Consider the parametric LCP $(q(\lambda) = b+\lambda b^*, M)$. In the process of solving this problem by the parametric LCP algorithm discussed above, let $y = (y_1, \dots, y_n)$, where $y_j \in \{w_j, z_j\}$ for each $j = 1$ to n, be a complementary basic vector obtained in some stage. Let $D_{.j}$ be the column vector associated with y_j in (5.1) for $j = 1$ to n. Let $[\underline{\lambda}, \overline{\lambda}]$ be the interval of feasibility of y. To find solutions for the parametric LCP $(q(\lambda), M)$ when $\lambda > \overline{\lambda}$, suppose we have to make a principal pivot step in position r. Let t_r be the complement of y_r and let $A_{.r}$ be the column associated with t_r in (5.1). So $A_{.r}$ is the complement of $D_{.r}$. Since the value of y_r in the solution in 5.1 is zero when $\lambda = \overline{\lambda}$, we have $q(\overline{\lambda}) \in \text{Pos}\{D_{.1}, \dots, D_{.r-1}, D_{.r+1}, \dots, D_{.n}\}$. Thus the portion of the straight line $\mathbf{L}$ in (5.1) corresponding to $\underline{\lambda} \leqq \lambda \leqq \overline{\lambda}$ lies in the complementary cone $\mathbf{K}_1 = \text{Pos}\{D_{.1}, \dots, D_{.n}\}$, and as λ increases through $\overline{\lambda}$, it leaves the cone $\mathbf{K}_1$ through its facet $\mathbf{F} = \text{Pos}\{D_{.1}, \dots, D_{.r-1}, D_{.r+1}, \dots, D_{.n}\}$. Let $\mathbf{H}$ denote the hyperplane in $\mathbf{R}^n$ which is the linear hull of $\{D_{.1}, \dots, D_{.r-1}, D_{.r+1}, \dots, D_{.n}\}$. Let $\overline{A}_{.r} = (\overline{a}_{1r}, \dots, \overline{a}_{nr})^T$ be the updated column associated with t_r. By Theorem 3.16, the hyperplane $\mathbf{H}$ strictly separates $D_{.r}$ and $A_{.r}$ iff $\overline{a}_{rr} < 0$. If $\overline{a}_{rr} < 0$, $q(\overline{\lambda})$ is on the common facet $\mathbf{F}$ of the complementary cones $\mathbf{K}_1$ and $\mathbf{K}_2 = \text{Pos}\{D_{.1}, \dots, D_{.r-1}, A_{.r}, D_{.r+1}, \dots, D_{.n}\}$. See Figure 5.1. As λ increases beyond $\overline{\lambda}$, the line $\mathbf{L}$ leaves the complementary cone $\mathbf{K}_1$ and enters the complementary cone $\mathbf{K}_2$ through their common facet $\mathbf{F}$.

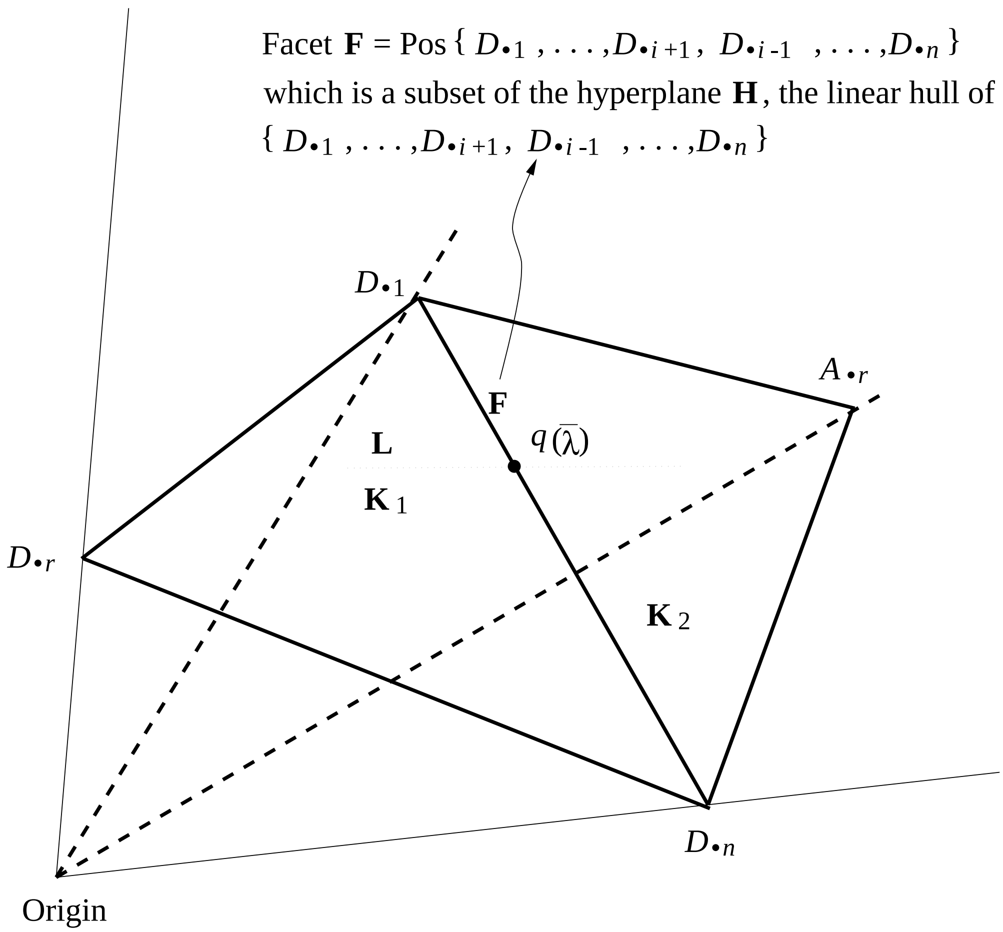

Figure 5.1 Situation when $\overline{a}_{rr} < 0$. As λ increases through $\overline{\lambda}$, the point $q(\lambda)$ travels along the straight line **L**, leaves the complementary cone $\mathbf{K}_1$ and enters the complementary cone $\mathbf{K}_2$, through their common facet **F**.

If M is a P-matrix, by the strict separation property discussed in Section 3.3, this situation occurs whenever Step 3 or 4 is carried out in the parametric LCP algorithm, and the algorithm finds the solutions of the parametric LCP for all values of the parameter λ.

If $\overline{a}_{rr} = 0$, $A_{.r}$ lies on the hyperplane **H** itself. If $\overline{a}_{rr} > 0$, $A_{.r}$ lies on the same side of the hyperplane **H** as $D_{.r}$. In either of these cases, as λ increases through $\overline{\lambda}$, the line **L** leaves both the complementary cones $\mathbf{K}_1$ and $\mathbf{K}_2$ and $(y_1, \ldots, y_{r-1}, t_r, y_{r+1}, \ldots, y_n)$ is not a complementary feasible basic vector for the parametric LCP $(q(\lambda), M)$ when $\lambda > \overline{\lambda}$. Hence if $\overline{a}_{rr} \geqq 0$, the parametric LCP algorithm is unable to find solutions of the parametric LCP $(q(\lambda), M)$ when λ increases beyond $\overline{\lambda}$.

Hence, geometrically, the parametric LCP algorithm discussed above can be interpreted as a walk along the straight line **L** crossing from one complementary cone in $\mathcal{C}(M)$ to an adjacent complement cone through their common facet.

Exercises

5.1 Let M be a given PSD matrix of order n, which is not PD. Discuss an approach for solving the parametric LCP $(q(\lambda) = b + \lambda b^*, M)$ for all values of λ for which it has a solution, and determining the range of values of λ for which it has no solution, based on the Graves' principal pivoting method of Section 4.2.

5.2 Suppose M is a copositive plus matrix and not a P-matrix. Discuss an approach for processing the parametric LCP (5.1) in this case, by the algorithm discussed above, using the complementary pivot algorithm to extend the value of λ whenever the pivot element in the parametric algorithm turns out to be nonnegative. Also prove that in this case, the set of all values of λ for which the parametric LCP (5.1) has a solution, is an interval.

5.1 PARAMETRIC CONVEX QUADRATIC PROGRAMMING

Here we consider a problem of the following form:

$$\begin{array}{lrl} \text{minimize} & Q_\lambda(x) = & (c + \lambda c^*)x + \frac{1}{2}x^T D x \\ \text{subject to} & Ax \geqq & b + \lambda b^* \\ & x \geqq & 0 \end{array} \tag{5.4}$$

where D is a symmetric PSD matrix of order n, and λ is a real valued parameter. The parameter λ in the right hand side constants vector in the constraints, and the linear part of the objective function, is the same. If $b^* = 0$, or $c^* = 0$, we get the special case of the problem in which the parameter appears in only the right hand side constants vector, or the linear part of the objective function, respectively. It is required to find an optimum solution of this problem, treating λ as a parameter, for all values of λ.

By the results in Chapter 1, this problem is equivalent to a parametric LCP $(q + \lambda q^*, M)$ where M is a PSD matrix. For the problem above, the data in the parametric LCP is given by

$$M = \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix}, \quad q = \begin{pmatrix} c^T \\ -b \end{pmatrix} \quad q^* = \begin{pmatrix} (c^*)^T \\ -b^* \end{pmatrix}. \tag{5.5}$$

We now discuss an algorithm for solving problems of this type. In preparing this section, I benefitted a lot from discussions with R. Saigal.

Algorithm for Parametric LCP $(q + \lambda q^*, M)$ When M is PSD

Initialization

Find a value for the parameter λ for which the system

$$\begin{aligned} w - Mz &= q + \lambda q^* \\ w,\ z &\geqq 0 \end{aligned} \tag{5.6}$$

has a feasible solution. Since M is PSD, by the results in Chapter 2, the LCP $(q + \lambda q^*, M)$ has a solution iff (5.6) has a feasible solution for that λ. Phase I of the parametric right hand side simplex method can be used to find a feasible solution for (5.6) (see Section 8.8 of [2.26]). When M, q, q^* are given by (5.5), if (5.6) is infeasible for a value of λ, by the results in Chapter 2, (5.4) does not have an optium solution for that λ (it is either infeasible, or $Q_\lambda(x)$ is unbounded below on the set of feasible solutions for it).

If there exists no value for λ for which (5.6) has a feasible solution, the parametric LCP $(q + \lambda q^*, M)$ does not have a solution for any λ, terminate. Otherwise, let λ_0 be a value of λ, for which (5.6) has a feasible solution (the parametric right hand side simplex algorithm, see Section 8.6 of [2.26], can in fact be used to determine the interval of values of λ for which (5.6) is feasible).

Now, find a complementary feasible basis for the LCP $(q + \lambda_0 q^*, M)$ with λ fixed equal to λ_0. The complementary pivot algorithm of Section 2.2 can be used for finding this. Since (5.6) is feasible when $\lambda = \lambda_0$ and M is PSD, by the results in Chapter 2, the complementary pivot algorithm applied on the LCP $(q + \lambda_0 q^*, M)$ will terminate with a complementary feasible basic vector for it, in a finite number of pivot steps, if the lexicographic minimum ratio rule is used to determine the dropping variable in each step. Let the complementary feasible basic vector be $y = (y_1, \ldots, y_n)$, (where $y_j \in \{w_j, z_j\}$ for each $j = 1$ to n), associated with the complementary basis B. Let $\overline{q}$, $\overline{q}^*$ be the updated right hand side constants vectors ($\overline{q} = B^{-1}q, \overline{q}^* = B^{-1}q^*$). Let

$$\begin{aligned} \lambda_1 &= -\infty, \text{ if } \overline{q}^* \leqq 0 \\ &= \text{ Maximum } \{-\overline{q}_i/\overline{q}_i^* : i \text{ such that } \overline{q}_i^* > 0\}, \text{ otherwise,} \\ \lambda_2 &= +\infty, \text{ if } \overline{q}^* \geqq 0 \\ &= \text{ Minimum } \{-\overline{q}_i/\overline{q}_i^* : i \text{ such that } \overline{q}_i^* < 0\}, \text{ otherwise.} \end{aligned}$$

Then, for all $\lambda_1 \leqq \lambda \leqq \lambda_2$, y is a complementary feasible basic vector for the parametric LCP $(q+\lambda q^*, M)$. This interval is nonempty since λ_0 is contained in it. In this interval, a complementary feasible solution for the parametric LCP is

$$\text{complement of } y \text{ is } 0$$

$$y = \overline{q} + \lambda \overline{q}^*.$$

Procedure to Increase the Value of λ

Suppose we have a complementary basic vector $y = (y_1, \ldots, y_n)$, where $y_j \in \{w_j, z_j\}$ for each $j = 1$ to n, corresponding to the complementary basis B, for which the upper characteristic value is $\overline{\lambda}$, which is finite. Here we discuss how to proceed to find complementary solutions of the parametric LCP when $\lambda > \overline{\lambda}$. Assume that B is lexico feasible for $\lambda = \overline{\lambda}$. Let $\beta = B^{-1}$, $\overline{q} = B^{-1}q$, $\overline{q}^* = B^{-1}q^*$. Then $(\overline{q}_i + \overline{\lambda}\overline{q}_i^*, \beta_{i.}) \succ 0$ for all $i = 1$ to n. Determine the i which attains the lexico minimum $\{-(\overline{q}_i, \beta_{i.})/(\overline{q}_i^*) : i$ such that $\overline{q}_i^* < 0\}$, and suppose it is p. Let the complement of the variable y_p be t_p. Suppose the updated column vector of t_p in the canonical tableau for

w	z	
I	$-M$	$q + \lambda q^*$

(5.7)

with respect to the complementary basic vector y be $(\overline{a}_{1p}, \ldots, \overline{a}_{np})^T$. Since M is PSD, by the results in Chapter 3, $\overline{a}_{pp} \leqq 0$.

If $\overline{a}_{pp} < 0$, performing a single principal pivot step in position p in the present complementary basic vector y, leads to a new complementary basic vector which will be feasible for some values of $\lambda > \overline{\lambda}$ under nondegeneracy. We repeat this whole process with that complementary basic vector.

If $\overline{a}_{pp} = 0$, to increase the value of λ beyond $\overline{\lambda}$, we enter into a special complementary pivot phase described below.

The Complementary Pivot Phase to Increase the Value of λ

We enter this phase when we obtain a complementary basic vector $y = (y_1, \ldots, y_n)$, where $y_j \in \{w_j, z_j\}$ for each $j = 1$ to n, with finite upper characteristic value $\overline{\lambda}$, and $\overline{a}_{pp} = 0$, as discussed above.

In the present canonical tableau, transfer the column of the parameter λ from the right hand side to the left hand side, and treat λ now as a variable. This leads to

Basic Variables	y	t	λ	
y	I	$\ldots$	$-\overline{q}^*$	$\overline{q}$

(5.8)

In this tableau, perform a pivot step in the column of λ, with row p as the pivot row, this is possible since $-\overline{q}_p^* > 0$. This leads to the following tableau.

Tableau 5.1

Basic Variables	$y_1 \dots y_{p-1}$ λ $y_{p+1} \dots y_n$	y_p $t_1 \dots t_n$	
y_1			
$\vdots$			
y_{p-1}	I	$\dots$	$\tilde{q}$
λ			
y_{p+1}			
$\vdots$			
y_n			

Tableau 5.1 is the canonical tableau with respect to the basic vector $(y_1, \dots, y_{p-1}, \lambda, y_{p+1}, \dots, y_n)$. As defined in Chapter 2, this is an ACBV (almost complementary basic vector). Here, λ plays the same role as the artificial variable z_0 in Chapter 2. There is one difference. In Chapter 2, z_0 was a nonnegative artificial variable, here λ is a variable which is a natural parameter, and it can take either negative or positive values.

From the manner in which Tableau 5.1 is obtained, it is clear that the value of λ in the basic solution corresponding to Tableau 5.1 is $\tilde{q}_p = -\overline{q}_p/\overline{q}^*_p = \overline{\lambda}$. Treat Tableau 5.1 as the original tableau for this phase. The word **basis** in this phase refers to the matrix of columns from Tableau 5.1, corresponding to the basic variables in any basic vector for Tableau 5.1. This phase requires moving among ACBVs in which λ will always be the pth basic variable. Let B be the basis corresponding to such an ACBV, and let $\hat{q} = B^{-1}\tilde{q}$, $\beta = B^{-1}$. This ACBV is said to be **feasible for this phase** if $\hat{q}_i \geqq 0$ for all $i \neq p$ and **lexico feasible for this phase** if $(\hat{q}_i, \beta_{i.}) \succ 0$ for all $i \neq p$. Let B be such a basis, let $\hat{q} = B^{-1}\tilde{q}$, $\beta = B^{-1}$ and suppose it is required to bring the column of a nonbasic variable, say x_s, into the basis B. Let $(\hat{a}_{1s}, \dots, \hat{a}_{ns})^T$ be the updated column of x_s (it is, B^{-1} (column of x_s in Tableau 5.1)). The **lexico minimum ratio test for this phase** determines the dropping variable to be the rth basic variable, where r is the i which attains the lexico minimum $\{(\hat{q}_i, \beta_{i.})/\hat{a}_{is} : i$ such that $i \in \{1, \dots, p-1, p+1, \dots, n\}$ and $\hat{a}_{is} > 0\}$. The **minimum ratio** for this pivot step, is defined to be $(\hat{q}_r/\hat{a}_{rs})$, it is always $\geqq 0$. The initial ACBV in Tableau 5.1 is lexico feasible in the sense defined here, and all the ACBVs obtained during this phase will have the same property.

Now, bring the variable t_p into the initial ACBV $(y_1, \dots, y_{p-1}, \lambda, y_{p+1}, \dots, y_n)$, determining the dropping variable by the lexico minimum ratio test as discussed above. Continue this phase using the **complementary pivot rule**, that is, the entering variable in any step, is always the complement of the dropping basic variable in the previous step. We prove below that the value of λ in the basic solution keeps on increasing in this phase.

At some stage, let $(\xi_1, \dots, \xi_{p-1}, \lambda, \xi_{p+1}, \dots, \xi_n)$ be the ACBV with the values of the basic variables in the corresponding BFS to be $\hat{q} = (\hat{q}_1, \dots, \hat{q}_n)^T$. So the value of λ in this solution is $\hat{q}_p$. Let v_s denote the entering variable into this ACBV, as determined by the complementary pivot rule. Let $(\hat{a}_{1s}, \dots, \hat{a}_{ns})^T$ be the pivot column (updated column of v_s), and let θ denote the minimum ratio, as defined above, for this step. We prove below that $\hat{a}_{ps} \leqq 0$. The solution

$$\begin{aligned} \xi_i &= \hat{q}_i - \alpha \hat{a}_{is}, i \in \{1, \dots, p-1, p+1, \dots, n\} \\ v_s &= \alpha \\ \text{all other variables} &= 0 \end{aligned}$$

is a complementary feasible solution of the original parametric LCP when $\lambda = \hat{q}_p - \alpha \hat{a}_{ps}$, for $0 \leqq \alpha \leqq \theta$. As the value of λ keeps on increasing during this phase, this process keeps getting solutions of the original parametric LCP for higher and higher values of λ, as the phase progesses.

This phase only terminates when an ACBV, say, $(\eta_1, \dots, \eta_{p-1}, \lambda, \eta_{p+1}, \dots, \eta_n)$ is reached satisfying the property that if ν denotes the entering variable into this ACBV, as determined by the complementary pivot rule, and $(a_1^*, \dots, a_n^*)^T$ is the pivot column (updated column of ν), then $a_i^* \leqq 0$ for all $i \in \{1, \dots, p-1, p+1, \dots, n\}$. This is similar to ray termination of Chapter 2. Let $q^* = (q_1^*, \dots, q_n^*)^T$ be the present updated right hand side constants vector. If $a_p^* < 0$, then the solution

$$\begin{aligned} \eta_i &= q_i^* - \alpha a_i^*, i \in \{1, \dots, p-1, p+1, \dots, n\} \\ \nu &= \alpha \\ \text{all other variables } &= 0 \end{aligned}$$

is a complementary solution of the original parametric LCP when $\lambda = q_p^* - \alpha a_p^*$, for all $\alpha \geqq 0$. In this case, this solution therefore, provides the solution of the parametric LCP for all $\lambda \geqq q_p^*$, terminate.

If $a_p^* = 0$ when this termination occurs, the original parametric LCP is infeasible whenever $\lambda > q_p^*$ (this fact is proved below), terminate.

Procedure to Decrease the Value of λ

Suppose we have a complementary basic vector $y = (y_1, \dots, y_n)$, for which the lower characteristic value is $\underline{\lambda}$, finite. Let $\beta = B^{-1}$ be the inverse of the complementary basis corresponding to y, and $\overline{q} = \beta q$, $\overline{q}^* = \beta q^*$. Assuming that y is lexico feasible for $\lambda = \underline{\lambda}$, we have $(\overline{q}_i + \underline{\lambda} \overline{q}_i^*, \beta_{i.}) \succ 0$ for all i. Determine the i that attains the lexico maximum $\{-(\overline{q}_i, \beta_{i.})/(\overline{q}_i^*) : i \text{ such that } \overline{q}_i^* > 0\}$, and suppose it is p. Let the updated column of the complement of y_p in the canonical tableau of (5.7) with respect to y be $(\overline{a}_{1p}, \dots, \overline{a}_{np})^T$. If $\overline{a}_{pp} < 0$, perform a single principal pivot step in position p in the present complementary basic vector y, and continue in the same way. If $\overline{a}_{pp} = 0$, to decrease λ below $\underline{\lambda}$, enter into a special complementary pivot phase. This phase begins with performing a pivot step in the column of λ in (5.8) with row p as the pivot row, to transform the column of λ in (5.8) into $-I_{.p}$ (the usual pivot step would transform the column of λ in (5.8) into $+I_{.p}$), leading to an ACBV as before. Except for this change, the complementary pivot procedure is carried out exactly as before. In all the canonical tableaus obtained in this phase, λ remains the pth basic variable, with its updated column as $-I_{.p}$. The value of λ keeps on decreasing as this phase progresses, and termination occurs when ray termination, as described earlier, occurs. During this procedure, the complementary solutions of the original parametric LCP for different values of λ are obtained using the same procedure as discussed earlier, from the basic solution of the system in Tableau 5.1 corresponding to the ACBV at each stage.

Proof of the Algorithm

Here we prove the claims made during the complementary pivot phase for increasing the value of λ.

Theorem 5.2 *Let $(\xi_1, \ldots, \xi_{p-1}, \lambda, \xi_{p+1}, \ldots, \xi_n)$ be an ACBV obtained during this phase. Let $\hat{q} = (\hat{q}_1, \ldots, \hat{q}_n)^T$ be the updated right hand side constants vector with respect to this ACBV. Let v_s denote the entering variable into this ACBV as determined by the complementary pivot rule. Let $(\hat{a}_{1s}, \ldots, \hat{a}_{1n})^T$ be the updated column of v_s. Then $\hat{a}_{ps} \leqq 0$, and the value of λ increases or remains unchanged when v_s enters this ACBV.*

Proof. We will first prove that $\hat{a}_{ps} \leqq 0$. The first ACBV in this phase was $(y_1, \ldots, y_{p-1}, \lambda, y_{p+1}, \ldots, y_n)$ and the entering variable into it is t_p. From the manner in which this phase was initiated, we know that the updated column of t_p in the canonical tableau of (5.7) with respect to y, $(\overline{a}_{1p}, \ldots, \overline{a}_{np})$, has its pth entry $\overline{a}_{pp} = 0$. Thus the pth entry in the column of t_p in Tableau 5.1 is also zero, and when t_p enters the ACBV in Tableau 5.1, no change occurs in its row p, which verifies the statement of this theorem for the initial ACBV in this phase. We will now show that it holds in all subsequent ACBVs obtained in this phase too.

Let $(\zeta_1, \ldots, \zeta_n)$ denote the ACBV just before the current ACBV $(\xi_1, \ldots, \xi_{p-1}, \lambda, \xi_{p+1}, \ldots, \xi_n)$. Suppose the statement of the theorem holds true in all steps of this phase until the ACBV ζ. We will now prove that this implies that the statement of this theorem must also hold for the complementary pivot step of bringing v_s into this ACBV $(\xi_1, \ldots, \xi_{p-1}, \lambda, \xi_{p+1}, \ldots, \xi_n)$.

Let u_s denote the complement of v_s. Since v_s is the entering variable chosen by the complementary pivot rule, u_s must have just dropped out of the basic vector ζ leading to the present basic vector $(\xi_1, \ldots, \xi_{p-1}, \lambda, \xi_{p+1}, \ldots, \xi_n)$. Let u_r denote the entering variable into the ACBV ζ that replaced u_s from it. Suppose the pivot row for entering u_r into ζ was row p' (so, u_s must have been the p'th basic variable in ζ). Let the updated entries in the canonical tableau of Tableau 5.1 with respect to the ACBV ζ, in rows p and p' be as given below.

Variable $\rightarrow$	ζ_1	$\ldots$	$\zeta_{p'} = u_s$	$\ldots$	$\zeta_p = \lambda$	ζ_{p+1}	$\ldots$	ζ_n	u_r	v_s	$\ldots$
row p'	0	$\ldots$	1	$\ldots$	0	0	$\ldots$	0	δ_1	δ_2	$\ldots$
row p	0	$\ldots$	0	$\ldots$	1	0	$\ldots$	0	δ_3	δ_4	$\ldots$

$\zeta = (\zeta_1, \ldots, \zeta_n)$ is an ACBV with $\zeta_p = \lambda$, and u_r is the entering variable into it chosen by the complementary pivot rule. These facts imply that $(\zeta_1, \ldots, \zeta_{p-1}, u_r, \zeta_{p+1}, \ldots, \zeta_n)$ becomes a complementary basic vector when the variables are properly ordered. It cannot be a basic vector unless $\delta_3 \neq 0$. So, $\delta_3 \neq 0$. Also since the statement of the theorem holds for the ACBV ζ, we have $\delta_3 \leqq 0$, so $\delta_3 < 0$. Also, since u_r is the entering variable into the ACBV ζ and row p' is the pivot row for this pivot step, we must have $\delta_1 > 0$. The pivot step in the column of u_r with δ_1 as the pivot element, transforms δ_4 into $\delta_4 - \frac{\delta_3 \delta_2}{\delta_1}$, by definition this is $\hat{a}_{ps}$, and we want to show that this is $\leqq 0$. As mentioned earlier, $(\zeta_1, \ldots, \zeta_{p-1}, u_r, \zeta_{p+1}, \ldots, \zeta_n)$ is a permutation of a complementary basic vector. So in the canonical tableau with respect to ζ, if we perform a pivot step in the column of u_r, with δ_3 as the pivot element (row p as the pivot row) and rearrange the rows and columns properly, we get the canonical tableau with respect to a complementary basic vector. This pivot step transforms the element δ_2 in the column of v_s into $\delta_2 - \frac{\delta_4 \delta_1}{\delta_3}$, this will be the entry in the column of v_s in row p', which is the row in which u_s is the basic variable. M is PSD, by the results of Chapter 3 every PPT of a PSD matrix is PSD, and by the results in Chapter 1 every diagonal entry in a PSD matrix is $\geqq 0$, these facts imply that this element $\delta_2 - \frac{\delta_4 \delta_1}{\delta_3} \leqq 0$. This, and $\delta_3 < 0$, $\delta_1 > 0$ established earlier imply that $\hat{a}_{ps} = \delta_4 - \frac{\delta_3 \delta_2}{\delta_1} \leqq 0$.

In all the pivot steps in this phase, the pivot elements are > 0, and all the updated right hand side constants with the possible exception of the pth, stay $\geqq 0$. These facts, and the fact that $\hat{a}_{ps} \leqq 0$ imply that when v_s enters the ACBV $(\xi_1, \ldots, \xi_{p-1}, \lambda, \xi_{p+1}, \ldots, \xi_n)$, the value of λ, the pth basic variable, either increases or stays the same (but never decreases).

Thus if the statement of the theorem holds for the ACBV ζ, it must hold for the ACBV $(\xi_1, \ldots, \xi_{p-1}, \lambda, \xi_{p+1}, \ldots, \xi_n)$ following it. We have already established the theorem for the initial ACBV in this phase. Hence, by induction, the theorem holds in all ACBVs obtained during this phase.

$\square$

So, the value of λ, the pth basic variable in the ACBV, increases during this phase. From the arguments used in Chapter 2, it is clear that the adjacency path of ACBVs in this phase continues unambiguously, and no ACBV can reappear. Since there are only a finite number of ACBVs, these facts imply that this phase must terminate with the special type of ray termination discussed here, after at most a finite number of steps.

We will now prove the claims made when ray termination occurs in this phase.

Theorem 5.3 *Let $(\eta_1, \ldots, \eta_{p-1}, \lambda, \eta_{p+1}, \ldots, \eta_n)$ be the terminal ACBV in the complementary pivot phase to increase λ. Let ν denote the entering variable into this ACBV chosen by the complementary pivot rule, and let $(a_1^*, \ldots, a_n^*)^T$ be the updated column of ν with respect to this ACBV. Let $q^* = (q_1^*, \ldots, q_n^*)^T$ be the updated right hand side constants vector with respect to this terminal ACBV in this phase. If $a_p^* = 0$, the original parametric LCP has no solution when $\lambda > q_p^*$.*

Proof. Let the complement of ν be u'. By the arguments used earlier, $(\eta_1, \ldots, \eta_{p-1}, u', \eta_{p+1}, \ldots, \eta_n)$ must be a permutation of a complementary basic vector. So, performing a pivot step in the canonical tableau with respect to the ACBV $(\eta_1, \ldots, \eta_{p-1}, \lambda, \eta_{p+1}, \ldots, \eta_n)$ with u' as the entering variable and row p as the pivot row, leads to a canonical tableau with respect to a complementary basic vector, with some rows and columns rearranged. Since $a^*_p = 0$, this pivot step would not alter the column vector of ν, and hence it remains as $(a^*_1, \ldots, a^*_n)^T \leqq 0$ with $a^*_p = 0$. M is PSD, and every PPT of a PSD matrix is PSD. These facts together with Result 1.6 imply that the updated row corresponding to u' in the canonical tableau (5.7) with respect to the complementary basic vector which is a permutation of $(\eta_1, \ldots, \eta_{p-1}, u', \eta_{p+1}, \ldots, \eta_n)$, has all nonnegative entries on the left hand side. When $\lambda > q^*_p$, the updated right hand side constant in this row will be < 0. This implies that the system (5.7) cannot have a nonnegative solution when $\lambda > q^*_p$, that is, that the original parametric LCP has no solution when $\lambda > q^*_p$.

□

5.2 Exercises

5.3 Let M, q, q^*, a be given matrices of orders $n \times n$, $n \times 1$, $n \times 1$, $n \times 1$ respectively. Assume that M is a P-matrix. Let $(w(\lambda), z(\lambda))$ be the solution of the parametric LCP $(q + \lambda q^*, M)$ as a function of λ. Let $\overline{\lambda}$ = maximum $\{\lambda : z(\lambda) \leqq a\}$. Also, let $\hat{\lambda}$ = maximum $\{\lambda : z(\alpha) \leqq a$, for all α satisfying $0 \leqq \alpha \leqq \lambda\}$. Discuss an efficient algorithm for find $\overline{\lambda}$, given M, q, q^*, a. Also, derive necessary and sufficient conditions on this data for $\hat{\lambda} = \overline{\lambda}$ to hold. (I. Kaneko [5.9] and O. De Donato and G. Maier [1.4]).

5.4 Let

$$M = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}, \qquad q(\lambda) = \begin{pmatrix} 1 & - \lambda \\ 2 & + \lambda \\ 3 & -2\lambda \end{pmatrix}.$$

Solve the parametric LCP $(q(\lambda), M)$ for all real values of λ.

5.5 Let $q = (-1, -2, -3)^T$ and M be the matrix given in Exercise 5.4. Solve the LCP (q, M) by Principal Pivoting Method I.

5.6 Prove that the value of z_0 (artificial variable) is strictly monotone decreasing in the complementary pivot method when applied on the LCP (q, M) associated with a P-matrix.

Prove that the same thing is true when the LCP (q, M) is one corresponding to an LP.

Is it also true when the LCP is one corresponding to a convex quadratic program in which the matrix D is PSD and not PD?

5.7 Consider the following problem

$$\begin{aligned}\text{minimize } z(x) &= cx + \alpha\sqrt{(1/2)(x^T Dx)}\\ \text{subject to } Ax &\geqq b\\ x &\geqq 0\end{aligned}$$

where D is a square symmetric PD matrix of order n, $\alpha > 0$, A is an $m \times n$ matrix and $b \in \mathbf{R}^m$. Let $\mathbf{K}$ denote the set of feasible solutions of this problem.

i) Show that $z(x)$ is a convex function which is a homogeneous function of degree 1.
ii) If $\alpha < \sqrt{2cD^{-1}c^T}$, prove that every optimum solution of this problem must be a boundary point of $\mathbf{K}$.
iii) If $0 \notin \mathbf{K}$ and if the problem has an optimum solution, prove that there exists a boundary point of $\mathbf{K}$ which is an optimum solution of the problem.
iv) Develop an efficient procedure for solving this problem.
v) Solve the problem

$$\begin{aligned}\text{minimize } & -x_1 - x_2 + \sqrt{(x_1^2 + x_2^2)/2}\\ \text{subject to } & -x_1 - 3x_2 \geqq -14\\ & -x_1 + x_2 \geqq -2\\ & x_1, x_2 \geqq 0\end{aligned}$$

using the method developed in (iv). (C. Sodini [5.15]).

5.8 Consider the following problem

$$\begin{aligned}\text{minimize } f(x) &= (c_0 + cx + (1/2)x^T Dx)/(d_0 + dx)^p\\ \text{subject to } Ax &\geqq b\\ x &\geqq 0\end{aligned}$$

where D is a square symmetric PD matrix of order n, p is 1 or 2 and $d_0 + dx > 0$ over $x \in \mathbf{K} = \{x : Ax \geqq b, x \geqq 0\}$. Develop an approach for solving this problem. (S. Schaible [5.14]; A. Cambini, L. Martein and C. Sodini [5.4]).

5.9 Consider the following problem

$$\begin{aligned}\text{minimize } Q_\alpha(x) &= cx + \frac{\alpha}{2}x^T Dx\\ \text{subject to } Ax &\geqq b\\ x &\geqq 0\end{aligned}$$

where D is a PSD matrix of order n, and α is a nonnegative parameter. It is required to solve this problem for all $\alpha \geqq 0$. Formulate this problem as a parametric LCP of the form $(q + \lambda q^*, M)$, $\lambda \geqq 0$, and discuss how to solve it.

Note 5.1 This problem arises in the study of portfolio models. The linear function $(-cx)$ may represent the expected yield, and the quadratic term $\frac{1}{2}x^T Dx$ may be the variance of the yield (the variance measures the extent of random fluctuation in the actual yield from the expected). $Q_\alpha(x)$ is a positive weighted combination of the two objectives which are to be minimized in this model.

5.10 If q is nondegenerate in the LCP (q, M) (i. e., if every solution (w, z) to the system of equations, $w - Mz = q$, makes at least n variables nonzero), prove that the number of solutions of the LCP (q, M) is finite.

5.11 Let $\mathcal{C}_1$ be the set of solutions of

$$\begin{aligned} w - Mz &= q \\ w, z &\geqq 0 \\ w_j z_j &= 0, \;\; j = 2 \text{ to } n. \end{aligned}$$

Prove that $\mathcal{C}_1$ is the union of disjoint paths in $\mathbf{R}^n$.

5.12 Consider the LCP (q, M). Define $\mathbf{S}(q) = \{(w, z) : (w, z)$ is a solution of the LCP $(q, M)\}$. Prove that if there exists a $q \in \mathbf{R}^n$ such that $\mathbf{S}(q)$ is a nonempty unbounded set, then $\mathbf{S}(0)$ contains a nonzero point, that is, the LCP $(0, M)$ has a nonzero solution.

5.3 References

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