Chapter 4

PRINCIPAL PIVOTING METHODS FOR LCP

In this chapter we discuss several methods for solving the LCP based on principal pivot steps. One common feature of these methods is that they do not introduce any artificial variable. These methods employ either single or double principal pivot steps, and are guaranteed to process LCPs associated with P-matrices or PSD-matrices or both. We consider the LCP (q, M) of order n, which is the following in tabular form.

w	z	q
I	$-M$	q

$$w, z \geqq 0, \qquad w^T z = 0 \tag{4.1}$$

4.1 PRINCIPAL PIVOTING METHOD I

This method is most useful for solving LCPs (q, M) in which M is a P-matrix. It only moves among complementary basic vectors for (4.1) which are infeasible, and terminates when a complementary feasible basic vector is obtained. It employs only single principal pivot steps. The initial complementary basic vector for starting the method is $w = (w_1, \ldots, w_n)$.

In this method, the variables may change signs several times during the algorithm, before a complementary solution is obtained in the final step.

In a general step, let $\overline{q} = (\overline{q}_1, \ldots, \overline{q}_n)^T$ be the updated right hand side constants vector in the present canonical tableau of (4.1). If $\overline{q} \geqq 0$, the present complementary basic vector is feasible and the present BFS of (4.1) is a solution of the LCP (q, M), terminate. If $\overline{q} \ngeq 0$, let

$$r = \text{ Maximum } \{i : i \text{ such that } \overline{q}_i < 0\} . \tag{4.2}$$

Make a single principal pivot step in position r, that is, replace the present basic variable in the complementary pair (w_r, z_r) by its complement. If this pivot step cannot be carried out because the pivot element is zero, the method is unable to continue further, and it terminates without being able to solve this LCP. Otherwise the pivot step is carried out and then the method moves to the next step.

Example 4.1

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \quad q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} .$$

The various canonical tableaus obtained in solving this LCP (q, M) by Principal Pivoting Method I are given below. In each tableau the pivot element is inside a box.

Basic Variable	w_1	w_2	w_3	z_1	z_2	z_3	q
w_1	1	0	0	−1	0	0	−1
w_2	0	1	0	−2	−1	0	−1
w_3	0	0	1	−2	−2	$\boxed{-1}$	−1
w_1	1	0	0	−1	0	0	−1
w_2	0	1	0	−2	$\boxed{-1}$	0	−1
z_3	0	0	−1	2	2	1	1
w_1	1	0	0	−1	0	0	−1
z_2	0	−1	0	2	1	0	1
z_3	0	2	$\boxed{-1}$	−2	0	1	−1
w_1	1	0	0	$\boxed{-1}$	0	0	−1
z_2	0	−1	0	2	1	0	1
w_3	0	−2	1	2	0	−1	1

Basic Variable	w_1	w_2	w_3	z_1	z_2	z_3	q
z_1	-1	0	0	1	0	0	1
z_2	2	-1	0	0	1	0	-1
w_3	2	-2	1	0	0	$\boxed{-1}$	-1
z_1	-1	0	0	1	0	0	1
z_2	2	$\boxed{-1}$	0	0	1	0	-1
z_3	-2	2	-1	0	0	1	1
z_1	-1	0	0	1	0	0	1
w_2	-2	1	0	0	-1	0	1
z_3	2	0	$\boxed{-1}$	0	2	1	-1
z_1	-1	0	0	1	0	0	1
w_2	-2	1	0	0	-1	0	1
w_3	-2	0	1	0	-2	-1	1

The solution of this LCP (q, M) is therefore $(w_1, w_2, w_3; z_1, z_2, z_3) = (0, 1, 1; 1, 0, 0)$.

Theorem 4.1 *Suppose M is a given P-matrix of order n. When Principal Pivoting Method I is applied on the LCP (q, M), it terminates with a complementary feasible basic vector for it in a finite number of pivot steps. Also, a complementary basic vector which appeared once in the course of this method never reappears in subsequent steps.*

Proof. The proof is by induction of n. Since M is a P-matrix, and all the pivot steps in the method are principal pivot steps, by Theorem 3.5, and Corollary 3.5 all the pivot steps required are possible, and the pivot element in all the pivot steps is strictly negative. If n is equal to 1, the theorem is easily verified to be true, and the method terminates after at most one pivot step. We now set up an induction hypothesis.

Induction Hypothesis: Suppose F is a P-matrix of order s and $p \in \mathbf{R}^s$. For $s \leqq n-1$, Principal Pivoting Method I applied on the LCP (p, F) solves it in a finite number of pivot steps without cycling.

We will now show that the induction hypothesis implies that Principal Pivoting Method I solves the LCP (q, M) of order n in a finite number of steps without cycling. Consider the principal subproblem of the LCP (q, M) in the variables $\omega = (w_2, \ldots, w_n)^T$, $\xi = (z_2, \ldots, z_n)^T$. If Principal Pivoting Method I is applied on this subproblem, by the induction hypothesis, it terminates in a finite number of pivot steps with a complementary feasible basic vector for it. Let $(y_2^l, y_3^l, \ldots, y_n^l)$, $l = 1$ to

k be the sequence of complementary basic vectors for this subproblem obtained under this method.

When Principal Pivoting Method I is applied on the original LCP (q, M), w_1 is a basic variable in the initial complementary basic vector w. The question of replacing w_1 from the basic vector will only arise for the first time when a complementary basic vector $(w_1, y_2, \ldots, y_n)$ associated with a complementary basis B_1 for (4.1) is reached, satisfying the property that if $\overline{q} = B_1^{-1}q$, then $\overline{q}_2 \geqq 0, \ldots, \overline{q}_n \geqq 0$ (i. e., $(y_2, \ldots, y_n)$ must be a complementary feasible basic vector for the principal subproblem in the variables ω, ξ). When such a complementary basis B_1 is obtained for the first time in the method, if $\overline{q}_1 < 0$, w_1 is replaced from the basic vector by z_1, and the method is continued. On the other hand if $\overline{q}_1 \geqq 0$, B_1 is a complementary feasible basis for (4.1) and the method terminates. Hence the first k basic vectors obtained when Principal Pivoting Method I is applied on (4.1) must be $(w_1, y_2^l, \ldots, y_n^l)$, $l = 1$ to k.

By Theorem 3.13, the LCP (q, M) has a unique solution. Suppose it is $(\hat{w}, \hat{z})$. We consider two possible cases separately.

Case 1: $\hat{z}_1 = 0$.

Since $(y_2^k, \ldots, y_n^k)$ is a complementary feasible basic vector for the principal subproblem in the variables ω, ξ, the hypothesis in this case, and Theorem 3.18 imply that $(w_1, y_2^k, \ldots, y_n^k)$ must be a complementary feasible basic vector for the LCP (q, M). Hence in this case the method solves the LCP (q, M) in a finite number of steps, without cycling.

Case 2: $\hat{z}_1 > 0$.

In this case Theorem 3.13 implies that every complementary basic vector of the form $(w_1, y_2, \ldots, y_n)$ must be an infeasible basic vector for (4.1). Let B_1 be the complementary basis for (4.1) corresponding to $(w_1, y_2^k, \ldots, y_n^k)$. If $\overline{q} = (\overline{q}_i) = B_1^{-1}q$, then $\overline{q}_1 < 0$, $\overline{q}_2 \geqq 0, \ldots, \overline{q}_n \geqq 0$, since $(y_2^k, \ldots, y_n^k)$ is a complementary feasible basic vector for the principal subproblem in the variables ω, ξ. Hence, the next basic vector obtained in Principal Pivoting Method I applied on the LCP (q, M) must be $(z_1, y_2^k, \ldots, y_n^k) = (u_1, \ldots, u_n) = u$, say. Let $v = (v_1, \ldots, v_n)$ where v_j is the complement of u_j, $j = 1$ to n. Let the canonical tableau of (4.1) with respect to u be

u	v	q
I	$-\widetilde{M}$	$\tilde{q}$

(4.3)

$\widetilde{M}$ is the PPT of M corresponding to the complementary basic vector u, and by Theorem 3.5, $\widetilde{M}$ is also a P-matrix. By our assumptions in this case, (4.3) is the system of equations in an LCP $(\tilde{q}, \widetilde{M})$ with (u_i, v_i) as the complementary pair of variables for $i = 1$ to n, which has a unique solution in which v_1 is zero. The subsequent complementary basic vectors obtained in the Principal Pivoting Method I applied on the LCP (q, M) are exactly those which will be obtained when the LCP $(\tilde{q}, \widetilde{M})$, which has the property that $v_1 = 0$ in its unique solution, is solved by the same method starting with u as the initial complementary basic vector. Applying the result established under Case 1 to (4.3), we conclude that when Principal Pivoting Method I is continued from (4.3), $u_1 = z_1$ remains as a basic variable, and after a finite number of principal pivot steps, a complementary feasible vector will be obtained. Also no cycling ever occurs.

This proves that under the induction hypothesis, the statement of the theorem holds for the LCP (q, M) which is of order n. The theorem has been verified for $n = 1$. Hence it holds for all n by induction.

□

When M is a P-matrix, all the pivot elements in pivot steps encountered under Principal Pivoting Method I applied on the LCP (q, M) will be strictly negative, by Corollary 3.7. If row r is the pivot row, just before this pivot step the updated right hand side constant in row r is strictly negative (and this is the bottom most row with this property at this stage) and just after this pivot step, the updated right hand side constant in row r becomes strictly positive.

The pivot row choice rule (4.2) in Principal Pivoting Method I is only one of the rules which guarantee finite termination when M is a P-matrix. Actually, let $(i_1, i_2, \dots, i_n)$ be any permutation of $(1, 2, \dots, n)$. Select this permutation at the beginning of the method arbitrarily, but keep it fixed during the method. Suppose, instead of selecting r as in (4.2), it is selected by the following rule:

$$s = \text{ Maximum } \{t : t \text{ such that } \overline{q}_{i_t} < 0\}, r = i_s \, . \tag{4.4}$$

The rule (4.4) selects row r as the last row in which the updated right hand side constant vector is strictly negative when the rows are listed in the fixed order $(i_1, i_2, \dots, i_n)$. The rule (4.4) becomes rule (4.2) if the permutation $(i_1, \dots, i_n)$ is $(1, 2, \dots, n)$. It can be verified that Principal Pivoting Method I with the rule (4.4) for selecting r, instead of (4.2), again solves the LCP (q, M) in a finite number of steps without cycling, if M is a P-matrix. The proof is very similar to the proof of Theorem 4.1. Instead of looking at the principal subproblem of the LCP (q, M) in the variables $((w_2, \dots, w_n);\ (z_2, \dots, z_n))$, look at the principal subproblem in the variables $((w_1, \dots, w_{i_1-1}, w_{i_1+1}, \dots, w_n);\ (z_1, \dots, z_{i_1-1}, z_{i_1+1}, \dots, z_n))$; and change the wording of the induction hypothesis to account for the new rule (4.4) of the choice of r in the method. Row i_1 plays the same role as row 1 did in the proof of Theorem 4.1.

Computational experience indicates that by the proper selection of the permutation of the rows $(i_1, \dots, i_n)$ and the use of (4.4) for choosing r in the Principal Pivoting Method I, its computational efficiency can be improved substantially. Verify that on the problem in Example 4.1 above, if the permutation of rows $(i_1, i_2, i_3) = (2, 3, 1)$ is used together with the rule (4.4) for the choice of r, Principal Pivoting Method I solves that problem after exactly one pivot step, whereas the original version of the method illustrated in Example 4.1 took seven pivot steps. However, no rules have been developed yet for the choice of the row permutation $(i_1, \dots, i_n)$ depending on the data in q, M, to guarantee that Principal Pivoting Method I solves the LCP (q, M) most efficiently.

Even with rule (4.2) for pivot row choice, the performance of Principal Pivoting Method I on LCPs (q, M) in which M is a positive definite matrix, was superior to other methods in computational tests. See, for example, reference [4.11] of M. M. Kostreva. The interesting fact is that when M is a P-matrix, Principal Pivoting Method I solves the LCP (q, M), whether q is degenerate or not, in a finite number of pivot steps without cycling, without the explicit use of any techniques for resolving degeneracy, like perturbation of the right hand side constants vector.

It is not necessary to calculate the canonical tableaus of (4.1) in each pivot step to implement Principal Pivoting Method I. Since it does not require the columns of the basis inverse other than the pivot column in any step, an implementation of this method using either the product form of the inverse, or the elimination form of the inverse would be the most convenient to use, when solving problems on a digital computer. Such an implementation improves the numerical stability and also the computational efficiency of the method.

When M is a general matrix (not a P-matrix), Principal Pivoting Method I may be forced to terminate without obtaining a complementary feasible basic vector for the LCP (q, M) if the required single principal pivot step cannot be performed in some step because the corresponding diagonal element in the PPT of M at that stage is zero. However, if M is a nondegenerate matrix (and not a P-matrix), all the required single principal pivot steps in Principal Pivoting Method I can always be carried out by Theorem 3.4. But in this case the pivot elements in some single principal pivot steps under the method may be strictly positive. In such a pivot step, the updated right hand side constant in the pivot row remains negative even after the pivot step, and if the method is continued after such a pivot step, the same complementary basic vector may reappear and cycling occurs. Thus, Principal Pivoting Method I seems to be most useful only for solving LCPs (q, M) where M is a P-matrix.

Comment 4.1 This method and the finiteness proof for it in the case when M is a P-matrix are taken from [4.14] of K. G. Murty.

4.1.1 Extension to an Algorithm for the Nonlinear Complementarity Problem

In [4.8] G. J. Habetler and M. M. Kostreva have extended the Principal Pivoting Method I into an algorithm for solving the nonlinear complementary problem (1.44). Let $f(x) = (f_1(x), \ldots, f_n(x))^T$, where each $f_i(x)$ is a real valued function defined on $\mathbf{R}^n$. f is said to be a P-**function**, if for all $x \neq y \in \mathbf{R}^n$, there exists an i such that $(x_i - y_i)(f_i(x) - f_i(y)) > 0$. Given f and $\mathbf{J} \subset \{1, 2, \ldots, n\}$ define $g^{\mathbf{J}}(x) = (g^{\mathbf{J}}_j(x))$, where

$$\begin{aligned} g^{\mathbf{J}}_j(x) &= x_j \quad \text{for } j \notin \mathbf{J} \\ &= f_j(x) \text{ for } j \in \mathbf{J}. \end{aligned} \tag{4.5}$$

The P-function f is said to be a **nondegenerate P-function**, if $g^{\mathbf{J}}(x)$ defined as in (4.5) is a function from $\mathbf{R}^n$ onto $\mathbf{R}^n$ for each subset $\mathbf{J} \subset \{1, \ldots, n\}$. If M is a given square matrix of order n, from Theorems 3.11, 3.12 it follows that the affine function $Mx + q$ is a nondegenerate P-function iff M is a P-matrix.

Consider the system of equations $g^{\mathbf{J}}(x) = 0$. If this system has a solution $\overline{x}$, then $\overline{x}$ is said to be a **complementary point** associated with the subset $\mathbf{J}$. A complementary point $\overline{x}$ clearly satisfies the complementary condition $x^T f(x) = 0$ in (1.44). If $f(x)$ is a nondegenerate P-function, it can be shown (see references [4.8, 4.12, 4.13]) that there exists a unique complementary point associated with any subset $\mathbf{J} \subset \{1, \ldots, n\}$; and that the NLCP (1.44) has a unique complementary feasible solution. The algorithm discussed here is guaranteed to solve the NLCP (1.44) when $f(x)$ is a nondegenerate P-function.

For any $\mathbf{J} \subset \{1, \ldots, n\}$, the solution of the system

$$g^{\mathbf{J}}(x) = 0 \tag{4.6}$$

can be found by iterative methods for solving systems of nonlinear equations such as Newton-Raphson method (see Section 2.7.2 and [10.33]). Newton-Raphson method begins with an initial point x^0 and generates a sequence of points by the iteration $x^{r+1} = x^r - (\nabla g^{\mathbf{J}}(x^r))^{-1} g^{\mathbf{J}}(x^r)$. We will denote the solution of (4.6) by the symbol $\overline{x}(\mathbf{J})$.

The Algorithm

Start with $\mathbf{J} = \emptyset$. In a general step suppose $\mathbf{J}$ is the current subset of $\{1, \ldots, n\}$. Find the associated complementary point $\overline{x}(\mathbf{J})$. If $\overline{x}(\mathbf{J}) + f(\overline{x}(\mathbf{J})) \geqq 0$, then the solution of NLCP (1.44) is $\overline{x}_j(\mathbf{J})$, terminate. If $\overline{x}(\mathbf{J}) + f(\overline{x}(\mathbf{J})) \not\geqq 0$, find $r = \min \cdot \{j : \overline{x}_j(\mathbf{J}) + f(\overline{x}(\mathbf{J})) < 0\}$. Define $\tilde{\mathbf{J}} = \mathbf{J} \setminus \{r\}$ if $r \in \mathbf{J}$, $\mathbf{J} \cup \{r\}$ otherwise, go to the next step with $\tilde{\mathbf{J}}$ as the new subset and continue.

In [4.8] G. J. Habetler and M. M. Kostreva have proved that if $f(x)$ is a nondegenerate P-function, this algorithm finds the unique solution of the NLCP (1.44) in a finite number of steps. Computational tests have indicated that this algorithm is quite efficient if implemented with an efficient and robust method for solving systems of nonlinear equations of the form (4.6).

4.1.2 Some Methods which Do not Work For LCP

Y. Bard's Method

A method similar to Principal Pivoting Method I was suggested by Y. Bard (see [4.1], pages 157 – 172). His method is the following: start with $w = (w_1, \ldots, w_n)$ as the initial complementary basic vector.

In a general step, let $\overline{q} = (\overline{q}_1, \ldots, \overline{q}_n)^T$ be the updated right hand side constants vector in the canonical tableau of (4.1) with respect to the current complementary basic vector. If $\overline{q} \geqq 0$, the present BFS of (4.1) is a solution of the LCP (q, M), terminate. If $\overline{q} \ngeqq 0$, let r be such that $\overline{q}_r$ = Minimum $\{\overline{q}_i : i$ such that $\overline{q}_i < 0\}$. If there is a tie, select an r among those tied, arbitrarily. Make a single principal pivot step in position r. If this pivot step cannot be carried out because the pivot element is zero, the method is unable to continue further, and it terminates without being able to solve this LCP. Otherwise the pivot step is carried out, and the method moves to the next step.

This method can cycle even when M is a P-matrix, as this following example constructed by L. Watson indicates.

Example 4.2

Let

$$M = \begin{pmatrix} 10 & 0 & -2 \\ 2 & 0.1 & -0.4 \\ 0 & 0.2 & 0.1 \end{pmatrix}, \quad q = \begin{pmatrix} 10 \\ 1 \\ -1 \end{pmatrix}.$$

It can be verified that M is a P-matrix. When this method is applied on the LCP (q, M), (4.1) with this data, the following complementary basic vectors are obtained.

Complementary Basic Vector	$\overline{q}^T$ = Transpose of the Updated Right Hand Side Constants Vector	r = Position of the Single Principle Pivot Step at this Stage
(w_1, w_2, w_3)	$(10, 1, -1)$	3
(w_1, w_2, z_3)	$(-10, -3, 10)$	1
(z_1, w_2, z_3)	$(1, -1, 10)$	2
(z_1, z_2, z_3)	$(-3, 10, -10)$	3
(z_1, z_2, w_3)	$(-1, 10, 1)$	1
(w_1, z_2, w_3)	$(10, -10, -3)$	2
(w_1, w_2, w_3)	$(10, 1, -1)$	3

Hence the method cycles, even though the choice of r in each step in this example was unambiguous. Let

$$M = \begin{pmatrix} 0.01 & -0.1 & 2 \\ -0.2 & 4.1 & -60 \\ -0.4 & -6.0 & 100 \end{pmatrix}, \quad q = \begin{pmatrix} 0.01 \\ -0.7 \\ 1.0 \end{pmatrix}.$$

Verify that M is PD. Apply Y. Bard's method on the LCP (q, M) with this data and verify that the method cycles, even though the choice of r in each step of the method is unambiguously determined.

The Least Recently Considered Pivot Row Choice Rule for Principal Pivoting Method I

Here the pivot row, r, is chosen by the following. Arrange the rows in any specific order at the beginning of the algorithm, say $1, 2, \dots, n$, and fix this order. In Step 1, choose the pivot row to be the first row with a negative right hand side constant, when the rows are examined in the specific order $1, 2, \dots, n$. To choose the pivot row in any subsequent step, identify which row was the pivot row in the previous step. Suppose it was row i. Now examine the rows in the specific order $i+1, \dots, n, 1, \dots, i-1$, and choose the first one with a negative updated right hand side constant as the pivot row.

This rule circles through the rows in the specific order beginning with the pivot row of the previous step, until it finds the first row eligible to be the pivot row in this step and chooses it. A rule similar to this for choosing the entering column in the primal simplex algorithm for linear programming problems has been found to make it significantly more efficient. Hence this rule was proposed for the pivot row choice in Principal Pivoting Method I, with the hope that it will be computationally more efficient. With this rule, the method does not work, unfortunately. Consider the LCP (q, M) in Example 4.2. When this method is applied on that problem, it can be verified that it goes through exactly the same pivot steps as in Example 4.2 and cycles.

A Block Pivoting Method for the Linear Complementarity Problem

Let M be a square matrix of order n. Consider the following method for solving the LCP (q, M). Start with any complementary basic vector for (4.1), say, $w = (w_1, \dots, w_n)$.

In a general step let $y = (y_1, \dots, y_n)$ be the present complementary basic vector, and let $\overline{q} = (\overline{q}_1, \dots, \overline{q}_n)$ be the updated right hand side constants vector in the canonical tableau of (4.1) with respect to y. If $\overline{q} \geqq 0$, the present BFS is a solution of the LCP (q, M), terminate. If $\overline{q} \not\geqq 0$, define the complementary vector of variables $u = (u_1, \dots, u_n)$ by

$$
\begin{aligned}
u_j &= y_j \text{ if } \overline{q}_j \geqq 0 \\
&= \text{ complement of } y_j, \text{ if } \overline{q}_j < 0\,.
\end{aligned}
$$

If u is not a complementary basic vector (i. e., if the complementary set of column vectors corresponding to u is linearly dependent), the method terminates without being able to solve this LCP. If u is a complementary basic vector, a block pivot is made to obtain the canonical tableau with respect to the new complementary basic vector u, and the method moves to the next step.

Unfortunately this method can cycle even when M is a P-matrix and q is nondegenerate, as illustrated by the following example constructed by L. Watson. Let:

$$
M = \begin{pmatrix} 1 & 0 & -2 \\ -2 & 1 & 4 \\ -4 & 2 & 9 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}.
$$

When this method is applied on the LCP (q, M) beginning with the basic vector $w = (w_1, w_2, w_3)$, we get the following sequence of basic vectors completing a cycle.

Complementary Basic Vector	$\bar{q}^T =$ Transpose of the Updated Right Hand Side Constants Vector
(w_1, w_2, w_3)	$(1, -1, -3)$
(w_1, z_2, z_3)	$(-1, -3, 1)$
(z_1, w_2, z_3)	$(-3, 1, -1)$
(w_1, w_2, w_3)	$(1, -1, -3)$

In the LCP (q, M) if M is a P-matrix, and q is a nondegenerate the results in Theorem 3.22 indicate that the $2^{\mathbf{n}}$ complementary basic vectors for the problem are in **one to one correspondence** with the $2^{\mathbf{n}}$, n dimensional vectors of $+$ and $-$ sign symbols (these are the signs of the components in the updated right hand side constants vector with respect to the complementary basic vector). The LCP (q, M) is equivalent to the problem of finding the complementary basic vector corresponding to the sign vector consisting of all "+" sign symbols, under this one to one correspondence. This gives the problem a combinatorial flavor. It may be possible to develop an efficient algorithm to solve the LCP (q, M) under these conditions, based on this result.

4.2 THE GRAVES' PRINCIPAL PIVOTING METHOD

We will now discuss a **principal pivoting method** for solving LCPs developed by Robert L. Graves in [4.7]. This method is useful for solving LCPs (q, M) in which M is PSD. Consider the LCP (q, M) where M is a given PSD matrix of order n, (4.1). This method deals only with complementary basic vectors for (4.1), beginning with $w = (w_1, \dots, w_n)$ as the initial complementary basic vector. It uses only single or double principal pivot steps. All the complementary basic vectors obtained in the method, excepting possibly the terminal one, will be infeasible. When a complementary feasible basic vector for (4.1) is obtained, the method terminates. In this method also, variables may change signs several times during the algorithm.

The method requires a nonsingular square matrix of order n, say B, all of whose rows are lexicopositive initially. Any nonsingular square matrix of order n, whose rows are lexicopositive, can be used as the matrix B in the method. Whenever any pivot steps are carried out on (4.1), the same row operations are also carried out on the matrix B. Even though the row vectors of B are lexicopositive initially, they may not possess this property subsequently, after one or more pivot steps. In our discussion of this method, **we will choose B to be I**, the identity matrix of order n. When B is chosen as I, the updated B at any stage of the method will be the matrix consisting of the columns of w in the canonical tableau of (4.1) at that stage, and clearly, this will be the inverse of the complementary basis at that stage. Thus choosing B to be I is very convenient, because, all the computations in the method can then be performed efficiently using the basis inverse.

Instead of choosing B as I, if it is choosen as some general nonsingular matrix of order n whose rows are lexicopositive, the method is operated in the same way as below, with the exception that $\beta_{i.}$ is to be replaced by the ith row of the update of the matrix B. In this general method, the lexicopositivity of B is required so that the statement of the corresponding version of Theorem 4.4 discussed below, holds in Step 1 of this general method. We will now describe the method with $B = I$.

The Graves' Principal Pivoting Method

The initial complementary basic vector is $w = (w_1, \dots, w_n)$. In a general step, let $y = (y_1, \dots, y_n)$, where $y_j \in \{w_j, z_j\}$ for each $j = 1$ to n, be the present complementary basic vector. Let $\beta = (\beta_{ij})$ be the inverse of the complementary basis corresponding to y. Let $\overline{q}$ be the present updated right hand side constants vector, that is, $\overline{q} = \beta q$. If $\overline{q} \geqq 0$, y is a complementary feasible basic vector for (4.1) and the present BFS is a solution of the LCP (q, M). Terminate. If $\overline{q} \ngeqq 0$, define the row vector $f = (f_1, \dots, f_n)$ in this step to be $f =$ lexico maximum $\{\beta_{i.}/\overline{q}_i : i \text{ such that } \overline{q}_i < 0\}$. Since $\beta = (\beta_{ij})$ is nonsingular, this lexico maximum is uniquely determined, and suppose it is attained by $i = r$. So $f = (\beta_{r.})/\overline{q}_r$. This is known as the **f-vector** in this step. Row r in the canonical tableau of (4.1) with respect to the present complementary basic vector, is known as the **crucial row** in this step. Let t_r denote the complement of y_r and let $A_{.r}$ be the column vector corresponding t_r in the original tableau (4.1). The updated column of t_r is $\overline{A}_{.r} = \beta A_{.r} = (\overline{a}_{1r}, \dots, \overline{a}_{nr})^T$, say. If $\overline{a}_{rr} \neq 0$, perform a single principal pivot step in position r in the present complementary basic vector y and go to the next step. If

$$\overline{a}_{rr} = 0, \text{ and } \overline{a}_{ir} \leqq 0 \text{ for all } i \tag{4.7}$$

under the assumption that M is PSD, (4.1) does not even have a nonnegative solution (this is proved in Theorem 4.2 below) and hence, the LCP (q, M) has no solutions. Terminate. If $\overline{a}_{rr} = 0$ and $\overline{a}_{ir} > 0$ for at least one i, find lexico maximum $\{(\beta_{i.} - \overline{q}_i(\beta_{r.}/\overline{q}_r))/\overline{a}_{ir} : i \text{ such that } \overline{a}_{ir} > 0\}$. Let s be the i which attains this lexico maximum (it is shown in Theorem 4.3 bleow, that this s is unique). Perform a double principal pivot step in positions r and s in the present complementary basic vector y (we show in Theorem 4.3 below that this is possible under the assumption that M is PSD), and go to the next step.

Example 4.3

Consider the following LCP associated with a PSD matrix.

w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	
1	0	0	0	−1	2	−1	1	−4
0	1	0	0	−2	0	2	−1	−4
0	0	1	0	1	−2	0	3	2
0	0	0	1	−2	1	−3	−3	1

$w_j, z_j \geqq 0$, for all j, $w_j z_j = 0$ for all j

We denote the f-row in the kth step by f^k. We denote the inverses of the various complementary bases obtained in the method as β^k, $k = 1, 2, \dots$.

The symbol $\overline{A}_{.j}$ represents the present updated column of the entering variable.

First Inverse Tableau

Basic Variable	$\beta^1 =$ Inverse of the Complementary Basis				Updated q
w_1	1	0	0	0	−4
w_2	0	1	0	0	−4
w_3	0	0	1	0	2
w_4	0	0	0	1	1

Step 1: The f-row in this step is lexico maximum $\{-(1,0,0,0)/4, -(0,1,0,0)/4\} = (0, -1/4, 0, 0)$. So $r = 2$ and row 2 is the crucial row. The present basic variable in the crucial row is w_2, its complement z_2 has the updated column vector $\overline{A}_{.2} = (2, 0, -2, 1)^T$. Since $\overline{a}_{22} = 0$, we compute lexico maximum $\{((1,0,0,0) - (-4)(0,-1/4,0,0))/2, ((0,0,0,1) - (0,-1/4,0,0))\}$ and this is attained by $s = 1$. So we carry out a double principal pivot step in positions 2, 1. This leads to

Second Inverse Tableau

Basic Variable	$\beta^2 =$ Inverse of the Complementary Basis				Updated q
z_1	0	$-1/2$	0	0	2
z_2	1/2	$-1/4$	0	0	-1
w_3	1	0	1	0	-2
w_4	$-1/2$	$-3/4$	0	1	6

Step 2: The f-vector here is lexico maximum $\{-(1/2, -1/4, 0, 0), -(1, 0, 1, 0)/2\} = (-1/2, 1/4, 0, 0)$. So $r = 2$ and the second row is the crucial row again. The present basic variable in the crucial row is z_2, its complement w_2 has the updated column $\beta^2 I_{.2} = (-1/2, -1/4, 0, -3/4)^T$. Since $\overline{a}_{22} = -1/4 \neq 0$, we perform a single principal pivot step in position 2. This leads to

Third Inverse Tableau

Basic Variable	$\beta^3 =$ Inverse of the Complementary Basis				Updated q
z_1	-1	0	0	0	4
w_2	-2	1	0	0	4
w_3	1	0	1	0	-2
w_4	-2	0	0	1	9

Step 3: From the third inverse tableau we get $f^3 =$ lexico maximum $\{-(1, 0, 1, 0)/2\} = (-1/2, 0, -1/2, 0)$. So $r = 3$ and the crucial row is row 3 in this step. The basic variable in the crucial row is w_3, and the updated column vector of its complement, z_3, is $\overline{A}_{.3} = \beta^3(-1, 2, 0, -3)^T = (1, 4, -1, -1)^T$. Since $\overline{a}_{33} = -1 \neq 0$, we have to carry out a single principal pivot in position 3 in this step. This leads to

Fourth Inverse Tableau

Basic Variable	$\beta^4 =$ Inverse of the Complementary Basis				Updated q
z_1	0	0	1	0	2
w_2	2	1	4	0	-4
z_3	-1	0	-1	0	2
w_4	-3	0	-1	1	11

Step 4: From the fourth inverse tableau we get $f^4 =$ lexico maximum $\{-(2,1,4,0)/4\}$ $= (-1/2, -1/4, -1, 0)$. $r = 2$ and row 2 is the crucial row. w_2 is the present basic variable in the crucial row, the updated column vector of its complement, z_2, is $\overline{A}_{.2} = \beta^4(2, 0, -2, 1)^T = (-2, -4, 0, -3)^T$. Since $\overline{a}_{22} = -4 \neq 0$, we do a single principal pivot in position 2. This leads to

Fifth Inverse Tableau

Basic Variable	$\beta^5 =$ Inverse of the Complementary Basis				Updated q
z_1	-1	$-1/2$	-1	0	4
z_2	$-1/2$	$-1/4$	-1	0	1
z_3	-1	0	-1	0	2
w_4	$-9/2$	$-3/4$	-4	1	14

Since the updated q vector is now nonnegative, (z_1, z_2, z_3, w_4) is a complementary feasible basic vector. The BFS: $(w_1, w_2, w_3, w_4; z_1, z_2, z_3, z_4) = (0, 0, 0, 14; 4, 1, 2, 0)$ is a solution of this LCP. Terminate.

Example 4.4

Consider the LCP for which the original tableau is given below (M can be verified to be a PSD matrix in this problem).

w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	q
1	0	0	0	−1	1	−1	−1	2
0	1	0	0	−1	−1	0	−2	0
0	0	1	0	1	0	−1	0	−2
0	0	0	1	1	2	0	0	−1

$w_j, z_j \geqq 0$, and $w_j z_j = 0$, for all j

Step 1: The initial complementary basic vector is (w_1, w_2, w_3, w_4). We compute $f^1 =$ lexico maximum $\{-(0,0,1,0)/2, -(0,0,0,1)\} = (0,0,0,-1)$. So $r = 4$, and the crucial row is row 4. w_4 is the present basic variable in the crucial row, and the updated column vector of its complement, z_4, is: $\overline{A}_{.4} = (-1,-2,0,0)^T$. $\overline{a}_{44} = 0$, and we find that $\overline{a}_{i4} \leqq 0$ for all i. So condition (4.7) is satisfied in this step. The method therefore terminates with the conclusion that the LCP has no solution. Actually, the constraint corresponding to the fourth row is $w_4 + z_1 + 2z_2 = -1$, which by itself has no nonnegative solution. This clearly implies that this LCP (q, M) has no solution.

Proof of the Method

Theorem 4.2 *If M is* PSD *and condition (4.7) is satisfied in some step of the Graves' principal pivoting method applied on (4.1), there exists no feasible solution to* $w - Mz = q$, $w \geqq 0$, $z \geqq 0$.

Proof. Let $y = (y_1, \ldots, y_n)$ where $y_i \in \{w_j, z_j\}$ for each $j = 1$ to n, be the complementary basic vector in the step in which condition (4.7) is satisfied.

Let $t = (t_1, \ldots, t_n)$ where t_j is the complement of y_j for $j = 1$ to n. Let the canonical tableau with respect to the complementary basic vector y be

y	t	
I	$\overline{A}$	$\overline{q}$

Let row r be the crucial row in this step. By (4.7), $\overline{a}_{rr} = 0$ and $\overline{a}_{ir} \leqq 0$ for all i. Since M is PSD, its PPT $-\overline{A}$ is also PSD, and hence by Result 1.6 $\overline{a}_{ir} + \overline{a}_{ri} = 0$ for all i. So $\overline{a}_{ri} \geqq 0$ for all i. So the equation corresponding to the crucial row, row r, in the present canonical tableau, is $y_r + \sum_{i=1}^{n} \overline{a}_{ri} t_i = \overline{q}_r$. Since $\overline{q}_r < 0$ (as row r is the crucial row) and $\overline{a}_{ri} \geqq 0$ for all i, this by itself has no nonnegative solution. This implies that there exists no (w, z) satisfying $w - Mz = q$, $w \geqq 0$, $z \geqq 0$.

∎

Theorem 4.3 *In some step of the Graves' principal pivoting method applied on (4.1), if the crucial row is row r, and a single principal pivot in position r cannot be carried out, and if condition (4.7) is not satisfied, then the position s is determined unambiguously. Also, if M is* PSD, *then a double principal pivot in positions r, s is possible in this step.*

Proof. Let y be the complementary basic vector in the step under discussion. Let $\beta = (\beta_{ij})$ be the inverse of the complementary basis associated with y. Let $\overline{q} = \beta q$. Let $-\overline{A}$ be the PPT of M corresponding to y. The hypothesis in the theorem implies that $\overline{a}_{rr} = 0$. Suppose $i = h$, k both tie for the lexico maximum for determining s. Then $(\beta_{h.} - \overline{q}_h(\beta_{r.}/\overline{q}_r))/\overline{a}_{hr} = \beta_{k.} - (\overline{q}_k(\beta_{r.}/\overline{q}_r))/\overline{a}_{kr}$, which is a contradiction to the nonsingularity of the basis inverse β. So s is determined unambiguously.

Now, let $\overline{A}_{.s}$ be the updated column vector associated with the complement of y_s. The double principal pivot step of replacing y_r, y_s in the complementary basic vector y by their complements, is possible iff the order two determinant $\begin{pmatrix} \overline{a}_{ss} & \overline{a}_{sr} \\ \overline{a}_{rs} & \overline{a}_{rr} \end{pmatrix} \neq 0$. Since $\overline{a}_{rr} = 0$, $\overline{a}_{sr} > 0$ in this case, and $\overline{a}_{rs} = -\overline{a}_{sr} \neq 0$, this order two determinant is nonzero. So the double principal pivot in positions r and s is possible in this step.

□

Theorem 4.4 *Let M be a* PSD *matrix. Let $\hat{\beta}$ be the inverse of the complementary basis, and $\hat{q}$ the updated right hand side constants vector, in some step of the Graves' principal pivoting method applied on (4.1). If row l is the crucial row in this step, then*

$$\hat{\beta}_{i.} \succ \hat{q}_i(\hat{\beta}_{l.}/\hat{q}_l) \quad \text{for all } i \neq l\,. \tag{4.8}$$

Proof. Since the method begins with w as the initial complementary basic vector, the inverse of the initial complementary basis is I, all of whose rows are lexicopositive. From this, and from the definition of the crucial row in Step 1 of the method, it can be verified that the statement of the theorem holds true in Step 1 of the method. We now show that if the statement of the theorem holds in a step, say step k, then it also holds in the next step $k+1$.

Suppose $\hat{\beta}$ is the inverse of the complementary basis and $\hat{q}$ the updated right hand side constants vector in step $k+1$ of the method, where $k \geqq 1$. In the previous step, step k, let y be the complementary basic vector, and let β be the inverse of the complementary basis corresponding to y. Let row r be the crucial row in step k. Let $\overline{q} = \beta q$, it is the updated right hand side constants vector in step k. Suppose the statement of the theorem holds true in step k, that is:

$$\beta_{i.} \succ \overline{q}_i(\beta_{r.}/\overline{q}_r) \quad \text{for all } i \neq r\,. \tag{4.9}$$

Let t_j be the complement of y_j for $j = 1$ to n and let $-\overline{A}$ be the PPT of M corresponding to the complementary basic vector y. Since M is PSD, by Theorem 3.10, $-\overline{A}$ is also a PSD matrix. So, by Results 1.5, 1.6 we have: $\overline{a}_{ii} \leqq 0$ for all i, and if $\overline{a}_{ii} = 0$, then $\overline{a}_{ij} + \overline{a}_{ji} = 0$ for all j. Since rows r, l are the crucial rows in steps k, $k+1$, we have $\overline{q}_r < 0$, $\hat{q}_l < 0$.

If the pivot step in step k is a single principal pivot step in position r, we have $\overline{a}_{rr} < 0$, $\hat{q}_r = \overline{q}_r/\overline{a}_{rr} > 0$ (which implies that $l \neq r$, by the above facts), $\hat{\beta}_{r.} = \beta_{r.}/\overline{a}_{rr}$; $\hat{\beta}_{i.} = \beta_{i.} - \beta_{r.}(\overline{a}_{ir}/\overline{a}_{rr})$, for $i \neq r$; $\hat{q}_i = \overline{q}_i - \overline{q}_r(\overline{a}_{ir}/\overline{a}_{rr})$, for $i \neq r$. From (4.9) we have $\beta_{i.}\overline{q}_r \prec \overline{q}_i\beta_{r.}$. This implies that for all $i \neq r$, $(\beta_{i.} - \beta_{r.}(\overline{a}_{ir}/\overline{a}_{rr}))\overline{q}_r \prec (\overline{q}_i - \overline{q}_r(\overline{a}_{ir}/\overline{a}_{rr}))\beta_{r.}$, that is, $\hat{\beta}_{i.}\overline{q}_r \prec \hat{q}_i\beta_{r.}$. Since $\overline{a}_{rr} < 0$, this implies that for all $i \neq r$, $\hat{\beta}_{i.}\overline{q}_r/\overline{a}_{rr} \succ \hat{q}_i\beta_{r.}/\overline{a}_{rr}$. So $\hat{\beta}_{i.}\hat{q}_r \succ \hat{q}_i\hat{\beta}_{r.}$, or, $\hat{\beta}_{i.} \succ \hat{q}_i(\hat{\beta}_{r.}/\hat{q}_r)$, since $\hat{q}_r > 0$, for all $i \neq r$. From this we get $(\hat{\beta}_{i.}/\hat{q}_i) \prec (\hat{\beta}_{r.}/\hat{q}_r)$ for all $i \neq r$ satisfying $\hat{q}_i < 0$. Putting $i = l$ in this (since $\hat{q}_l < 0$) we get $(\hat{\beta}_{l.}/\hat{q}_l) \prec (\hat{\beta}_{r.}/\hat{q}_r)$. This and the previously proved statement that $\hat{\beta}_{i.} \succ \hat{q}_i(\hat{\beta}_{r.}/\hat{q}_r)$ together imply (4.8) for all $i \neq l$ such that $\hat{q}_i \geqq 0$. For $i \neq l$ such that $\hat{q}_i < 0$, (4.8) holds by the definition of the crucial row in step $k+1$. Thus, in this case, (4.8) holds in step $k+1$ if it holds in step k.

If the pivot in step k is a double principal pivot step in positions r, s, we have $\overline{q}_r < 0$, $\overline{a}_{rr} = 0$, $\overline{a}_{sr} > 0$, $\overline{a}_{rs} = -\overline{a}_{sr} < 0$. It can be verified that this pivot step yields

$$\begin{aligned}
\hat{\beta}_{r.} &= (\beta_{s.} - \beta_{r.}(\overline{a}_{ss}/\overline{a}_{rs}))/\overline{a}_{sr}\,,\ \hat{\beta}_{s.} = \beta_{r.}/\overline{a}_{rs} \\
\hat{\beta}_{i.} &= \beta_{i.} - \beta_{r.}(\overline{a}_{is}/\overline{a}_{rs}) - (\beta_{s.} - \beta_{r.}(\overline{a}_{ss}/\overline{a}_{rs}))(\overline{a}_{ir}/\overline{a}_{sr})\,,\ \text{for all } i \neq r, s \\
\hat{q}_r &= (\overline{q}_s - \overline{q}_r(\overline{a}_{ss}/\overline{a}_{rs}))/\overline{a}_{sr}\,, \hat{q}_s = \overline{q}_r/\overline{a}_{rs} \\
\hat{q}_i &= \overline{q}_i - \overline{q}_r(\overline{a}_{is}/\overline{a}_{rs}) - (\overline{q}_s - \overline{q}_r(\overline{a}_{ss}/\overline{a}_{rs}))(\overline{a}_{ir}/\overline{a}_{sr})\,,\ \text{for all } i \neq r, s\,.
\end{aligned} \tag{4.10}$$

We will now prove that, for $i \neq s$:

$$\hat{\beta}_{i.} \succ \hat{q}_i(\beta_{r.}/\overline{q}_r)\,. \tag{4.11}$$

First consider the case where $i \neq r$ or s. Substituting for $\hat{\beta}_i$, $\hat{q}_i$ and cancelling common terms, we verify that if $\overline{a}_{ir} \neq 0$,

$$\hat{\beta}_{i.} - \hat{q}_i\Big(\frac{\beta_{r.}}{\overline{q}_r}\Big) = \overline{a}_{ir}\left[\frac{1}{\overline{a}_{ir}}\Big(\beta_{i.} - \overline{q}_i\Big(\frac{\beta_{r.}}{\overline{q}_r}\Big)\Big) - \frac{1}{\overline{a}_{sr}}\Big(\beta_{s.} - \overline{q}_s\Big(\frac{\beta_{r.}}{\overline{q}_r}\Big)\Big)\right]\,. \tag{4.12}$$

If $\overline{a}_{ir} < 0$, from the choice of s and the fact that $\overline{a}_{sr} > 0$, we conclude that the right hand side of (4.12) is lexicopositive and hence (4.11) holds. On the other hand if $\overline{a}_{ir} > 0$, then from the choice of s we conclude that the right hand side of (4.12) is lexicopositive, and hence again (4.11) holds. If $\overline{a}_{ir} = 0$, from (4.10) we have $\hat{\beta}_{i.} - \hat{q}_i(\beta_{r.}/\overline{q}_r) = \beta_{i.} - \overline{q}_i(\beta_{r.}/\overline{q}_r)$ and by (4.9) this implies that (4.11) holds in this case too. So (4.11) holds for all $i \neq r, s$. Now consider $i = r$. From (4.10) we verify that $\hat{\beta}_{r.} - \hat{q}_r(\beta_{r.}/\overline{q}_r) = (\beta_{s.} - \overline{q}_s(\beta_{r.}/\overline{q}_r))/\overline{a}_{sr} \succ 0$ from (4.9) and the fact that $\overline{a}_{sr} > 0$. So (4.11) holds for $i \neq s$. Since l is the crucial row in step $k+1$, and $\hat{q}_s > 0$, we know that $l \neq s$. So from (4.11) we have $\hat{\beta}_{l.} \succ \hat{q}_l(\beta_{r.}/\overline{q}_r)$ and since $\hat{q}_l < 0$, this yields $(\hat{\beta}_{l.}/\hat{q}_l) \prec (\beta_{r.}/\overline{q}_r)$. Using this in (4.9) we get $\hat{\beta}_{i.} \succ \hat{q}_i(\hat{\beta}_{l.}/\hat{q}_l)$ for all i such that $i \neq s$ and $\hat{q}_i > 0$, which yields (4.8) for this i. If i is such that $\hat{q}_i < 0$, (4.8) follows from the choice of the crucial row in step $k+1$, since row l is the crucial row in step $k+1$. If i is such that $i \neq s$ and $\hat{q}_i = 0$, (4.8) follows from (4.11). If $i = s$, from (4.10) we conlcude $(\beta_{s.}/\hat{q}_s) = (\beta_{r.}/\overline{q}_r)$. We have already seen above that $(\beta_{r.}/\overline{q}_r) \succ (\hat{\beta}_{l.}/\hat{q}_l)$. So $(\hat{\beta}_{s.}/\hat{q}_s) \succ (\hat{\beta}_{l.}/\hat{q}_l)$ and since $\hat{q}_s > 0$ this implies (4.8) for $i = s$.

Thus whether the pivot step in step k is a single or double principal pivot step, if the statement of this theorem holds in step k, it holds in step $k+1$. We already verified that the statement of the theorem holds in step 1. Hence it holds in all steps of the method.

□

Theorem 4.5 *The f-vector undergoes a strict lexico-decrease in each step of the method, when applied on the LCP (q, M) where M is* PSD.

Proof. We consider a step in the method, say step k. As in the proof of Theorem 4.4, let β, $\hat{\beta}$ denote the inverse of the complementary bases; and let $\overline{q}$, $\hat{q}$ denote the updated right hand side constant vectors, in steps k, $k+1$ respectively. Let rows r, l be the crucial rows; and let f, $\hat{f}$ denote the f-vectors in steps k, $k+1$ respectively. We wish to prove that $\hat{f} \prec f$. From the definition of the crucial row we have: $f = \beta_{r.}/\overline{q}_r$, $\hat{f} = \hat{\beta}_{l.}/\hat{q}_l$. If the pivot in step k is a single principal pivot step, we have already shown in the proof of Theorem 4.4 that $(\hat{\beta}_{l.}/\hat{q}_l) \prec (\hat{\beta}_{r.}/\hat{q}_r) = (\beta_{r.}/\overline{q}_r)$ which implies that $\hat{f} \prec f$. If the pivot in step k is a double principal pivot step, we have already shown in the proof of Theorem 4.4 that $(\hat{\beta}_{l.}/\hat{q}_l) \prec (\beta_{r.}/\overline{q}_r)$ which implies that $\hat{f} \prec f$. So the f-vector undergoes a strict lexico decrease as the algorithm moves from step k to step $k+1$. So it undergoes a strict lexico decrease in each step of the method.

□

Theorem 4.6 *When M is* PSD, *the Graves' principal pivoting method either finds a solution of the LCP (q, M) or determines that it has no solution, in a finite number of steps.*

Proof. Each complementary basic vector for (4.1) corresponds to a unique f-vector. In each step of the method, if it does not terminate by either finding a complementary feasible basic vector, or by determining that the LCP (q, M) has no solution, the f-vector undergoes a strict lexico decrease, by Theorem 4.5. Hence in each step of the method, a new complementary basic vector is obtained, thus a complementary basic vector obtained in a step of the method, cannot reappear later on. Since there are at most $2^{\mathbf{n}}$-complementary basic vectors for (4.1), the method must terminate by either finding a complementary feasible basic vector (the BFS of (4.1) corresponding to which is a solution of the LCP (q, M)) or by determining that (4.1) does not even have a nonnegative solution, after at most $2^{\mathbf{n}}$ steps.

□

The proof of finite convergence of this method is quite novel, and is based on the fact that the f-vector undergoes a strict lexico decrease in each step. There is no objective function in LCPs and the f-vector is really extraneous to the problem, and yet, since the method guarantees that it undergoes a strict lexico decrease in each step, the method must terminate in a finite number of steps, and the only ways the method can terminate is by either finding a solution of the LCP or by determining that the LCP has no solution.

Theorem 4.7 *If the Graves' principal pivoting method is applied on the LCP (q, M) where M is a P-matrix, then the following statements hold:*

i) All pivot steps will be single principal pivot steps.
ii) In each step the pivot element is always strictly negative.
iii) The method terminates with a solution of the LCP in a finite number of steps without cycling.

Proof. (i) and (ii) follow from Corollary 3.5. It can be verified that the proof of Theorem 4.4 holds in this case too, and hence, the conclusion of Theorems 4.5, 4.6 remain valid here also. This implies (iii).

□

Thus the principal pivoting method discussed above can be applied to process LCPs (q, M) when M is either a PSD matrix or a P-matrix. However, when M is a P-matrix, Principal Pivoting Method I discussed in Section 4.1 will probably be much more efficient since it does not require the rows of the explicit basis inverse, or the determination of the lexico maximum of a set of row vectors in each step. The Graves' principal pivoting method has the advantage of processing LCPs (q, M) which M in PSD and not PD, and Principal Pivoting Method I may not be able to process these problems.

Exercises

4.1 Relationship of the Graves' Principal Pivoting Method to the Simplex Algorithm. Consider the LP (1.9) which can be written as

$$\begin{array}{ll} \text{Minimize} & cx \\ \text{subject to} & Ax - v = b \\ & x \geqq 0, v \geqq 0 \end{array} \tag{4.13}$$

where A is a matrix of order $m \times n$, $v = (v_1, \ldots, v_m)^T$, and $-b \geqq 0$. So v is a feasible basic vector for (4.13). The LCP corresponding to this LP is (q, M) with q, M given as in (1.10). Suppose the Graves' principal pivoting method is applied on (4.13). Then prove the following:

(i) All the pivots steps will be double principal pivot steps.
(ii) The columns of the PPT of M obtained in any step can be rearranged so that it has the structure

$$M' = \begin{pmatrix} 0 & -A'^T \\ A' & 0 \end{pmatrix} .$$

(iii) The rows of the inverse of the basis at the end of each step can be rearranged so that it has the following structure:

$$\beta = \begin{pmatrix} \beta^1 & 0 \\ 0 & \beta^2 \end{pmatrix}$$

where β^1, β^2 square nonsingular matrix of orders n and m respectively.

(iv) If $(\overline{c}, -\overline{b}^T)^T$ is the updated right hand side constants vector in any step, then $-\overline{b}$ is nonnegative.

(v) The sequence of basic solutions obtained in the Graves' principal pivoting method applied on (4.13) can be interpreted as the sequence of primal feasible and dual basic solutions obtained in the various steps of the primal simplex algorithm using the lexico minimum ratio rule for pivot row choice in each step, applied on the LP (4.13) beginning with the primal feasible basic vector v (R. L. Graves [4.7]).

4.2 Consider the quadratic program (1.11) discussed in Section 1.3. If $Q(x)$ is a convex function on $\mathbf{R}^n$, prove that the LCP (1.19) corresponding to it, is an LCP (q, M) in which the matrix M is PSD, and so it can be processed by the Graves' Principal Pivoting Method.

4.3 DANTZIG-COTTLE PRINCIPAL PIVOTING METHOD

This method due to G. B. Dantzig and R. W. Cottle [4.5, 4.6] pre-dates the other principal pivoting methods discussed so far, and evolved from a quadratic programming algorithm of P. Wolfe [4.18] who seems to be the first to use a type of complementary pivot choice rule. The method is useful for processing LCPs (q, M) in which M is either a P-matrix or a PSD matrix. The method goes through a sequence of what are called **major cycles**. Each major cycle begins with a complementary basic vector and ends with a complementary basic vector. Intermediate basic vectors in a major cycle are almost complementary basic vectors of the type discussed in Section 2.4. No artificial variable is introduced, but the original problem variables may take negative or nonnegative values during the method. When a nonnegative solution is obtained, it will be a complementary feasible solution of the LCP (q, M) and the method terminates. Once a variable becomes nonnegative in this method, it remains nonnegative in all subsequent steps (this property distinguishes this method from the other principal pivoting methods discussed so far). Also, if M is a P-matrix or a PD matrix, once a component of the updated q becomes nonnegative in this method, that particular component will remain nonnegative in all future updated qs. Each major cycle makes at least one more variable nonnegative. So there can be at most n major cycles when the method is applied to solve an LCP of order n. The first major cycle begins with $w = (w_j)$ as the initial complementary basic vector.

If q is nondegenerate, each component of the updated q remains nonzero throughout and there will never be any ties for the **blocking variable** (this term is defined below) in each step of any major cycle, thus identifying the blocking variable uniquely and unambiuously in every step. If q is degenerate, there may be ties for the blocking variable. However, as discussed in Section 2.2.8, in this case q can be perturbed to become nondegenerate, treating the perturbation parameter to be positive and small without giving any specific value to it. This requires the use of the lexico minimum ratio test in place of the usual minimum ratio test, whenever it is used, right from the beginning, and this again guarantees that the blocking variable is identified uniquely and unambiguously in each step. If the method can be proved to process the LCP (q, M) in a finite number of steps when q is nondegenerate, using arguments similar to those in Section 2.2.8 it can be proved that it will process it in a finite number of steps even when q is degenerate, if this lexico minimum ratio test is used in place of the minimum ratio test in each step. Because of this, without any loss of generality, we assume that q is nondegenerate, in the description of the method given below.

Case 1: M is a P-Matrix.

The first major cycle begins with $w = (w_1, \ldots, w_n)$ as the initial complementary basic vector.

Let $y = (y_1, \ldots, y_n)$ where $y_j \in \{w_j, z_j\}$ for $j = 1$ to n, be the initial complementary basic vector at the beginning of a major cycle. For $j = 1$ to n, let t_j be the complement of y_j. Let the canonical tableau of (4.1) with respect to y be

basic vector	y	t	
y	I	$-\overline{M}$	$\overline{q}$

$t = 0$ in the current solution, $y = \overline{q}$ (4.14)

If $\overline{q} \geqq 0$, y is a complementary feasible basic vector for the LCP (q, M) and we terminate. Otherwise select an r such that $\overline{q}_r < 0$. y_r will be called the **distinguished variable** in this major cycle. We try to make y_r increase from its present negative value in the solution, to zero, without allowing any variable already nonnegative to become negative. For this, we increase t_r from zero to a λ say. This leads to the new solution

$$\begin{aligned} y_i &= \overline{q}_i + \lambda \overline{m}_{ir}\,,\ i = 1 \text{ to } n \\ t_r &= \lambda\,,\ \text{all other } t_j = 0. \end{aligned} \tag{4.15}$$

Since M is a P-matrix, by Theorem 3.5, $\overline{m}_{rr} > 0$. Hence, in (4.15), the value of y_r increases as λ increases. So, in this role, t_r is called the **driving variable**. The increase in the value of the driving variable must stop as soon as a positive basic variable decreases to zero, or the distinguished variable increases to zero. The variable which thus limits the increase of the driving variable is called the **blocking variable**. To identify the blocking variable, find minimum $\{(\overline{q}_r/(-\overline{m}_{rr})); (\overline{q}_i/(-\overline{m}_{ir}))$, for all i such that $\overline{q}_i \geqq 0$ and $(-\overline{m}_{ir}) > 0\}$. Suppose this minimum is attained by $i = s$ (if there is a tie for this s, the lexico minimum ratio rule as in Sections 2.2.7, 2.2.8 should be used to break the tie, as discussed above).

If $s = r$, a principal pivot step in position r is carried out in (4.14), this leads to a complementary basic solution in which y_r is positive, and the method moves to the next major cycle with it.

If $s \neq r$, perform a pivot in (4.14) replacing the basic variable y_s by t_r, a nonprincipal pivot. The new basic vector obtained is almost complementary (as defined in Section 2.4), both the distinguished variable y_s and its complement are basic variables in it, both the blocking variable y_r and its complement are nonbasic. Let $-\overline{\overline{m}}_{is}$, $i = 1$ to n, be the entries in the updated column of t_s after this pivot step. Clearly $-\overline{\overline{m}}_{ss} = -\overline{m}_{ss}/(-\overline{m}_{sr}) < 0$ since $\overline{m}_{ss} > 0$ (since M is a P-matrix) and $(-\overline{m}_{sr}) > 0$ (by the choice of the blocking variable), and $-\overline{\overline{m}}_{rs} = -\overline{m}_{rs} + \overline{m}_{rr}\overline{m}_{ss}/\overline{m}_{sr} < 0$ since $\overline{m}_{sr} < 0$ (by choice of the blocking variable) and $\overline{m}_{rr}\overline{m}_{ss} - \overline{m}_{sr}\overline{m}_{rs} > 0$ (this is the principal subdeterminant of $\overline{M}$ corresponding to the subset $\{s, r\}$ which is positive since $\overline{M}$ is a P-matrix, being a PPT of the P-matrix M). The pivot step has left the distinguished variable basic at a negative value. The next variable to enter the basis, that is, the next driving variable, is the complement of the blocking variable which just became nonbasic; it is t_s here. Since we have shown that $(-\overline{\overline{m}}_{ss}) < 0$, $(-\overline{\overline{m}}_{rs}) < 0$ above, increasing the value of the new driving variable results in the continuing increase of both the distinguished variable and its complement. The increase of the new driving variable is also governed by the same rules as above. Since the value of the distinguished variable has been shown to increase, it is potentially a blocking variable, and hence a blocking variable exists again. Using the properties of P-matrices discussed in Chapter 3, it can be verified that all these properties continue to hold when the major cycle is continued with the same rules. A sequence of almost complementary basic vectors is obtained in the process, which can only terminate when the distinguished variable is driven up to zero, at which time it is the blocking variable, and the corresponding pivot leads to a complementary basic vector. Since the distinguished variable and its complement increase strictly from one pivot step to the next, no basis can be repeated, and hence the sequence is finite, as there are only a finite number of almost complementary basic vectors. The finiteness of the overall method follows since there are at most n major cycles (the number of negative variables decreases by at least one in each major cycle).

In this case it can be verified that once the entry in a row in an updated q becomes nonnegative, it stays nonnegative in all subsequent steps.

Case 2: M is a PSD Matrix, but not a P-Matrix

In this case it is possible that the system $w - Mz = q$, $w, z \geqq 0$ is not even feasible, and the method should be able to detect this possibility. As before let (4.14) be the canonical tableau at the beginning of a major cycle. Select the distinguished variable as in Case 1 to be the basic variable in a row in which the updated right hand side constant is negative, say y_r. Since M is PSD, its PPT $\overline{M}$ is also PSD by Theorem 3.10 and hence its diagonal entries are all nonnegative by Result 1.5. So $\overline{m}_{rr} \geqq 0$, and could be zero here.

Suppose $\overline{m}_{rr} = 0$. In addition, if $(-\overline{m}_{ir}) \leqq 0$ for all i, Result 1.6 implies that $(-\overline{m}_{rj}) \geqq 0$ for all j (since $\overline{M}$ is PSD and $\overline{m}_{rr} = 0$ we will have $\overline{m}_{ir} + \overline{m}_{ri} = 0$ for all i). The equation corresponding to the updated rth row is

$$y_r + \sum_{j=1}^{n} (-\overline{m}_{rj}) t_j = \overline{q}_r \, . \tag{4.16}$$

Under these conditions ($\overline{q}_r < 0$, $-\overline{m}_{rj} \geqq 0$ for all j), (4.16) does not even have a nonnegative solution, which implies that "$w - Mz = q$, $w \geqq 0$, $z \geqq 0$" has no feasible solution. So under these conditions the LCP (q, M) has no solution.

If $\overline{m}_{rr} = 0$, and the infeasibility condition ($-\overline{m}_{ir} \leqq 0$ for all i) is not satisfied; as in Case 1, we increase the value of the driving variable t_r from zero. However, since $\overline{m}_{rr} = 0$, it has no effect on the negative value of the distinguished variable. In addition, if $-\overline{m}_{ir} \leqq 0$ for all i satisfying $\overline{q}_i \geqq 0$, the increase in the value of the driving variable t_r, makes no nonnegative basic variable decrease. But under these conditions $-\overline{m}_{ir} > 0$ for at lest one i satisfying $\overline{q}_i < 0$, and the value of this ith basic variable decreases further from its present negative value as the value of the driving variable is increased. So there is no blocking variable in the sense discussed under Case 1. Also, under these conditions, since there is at least one $\overline{m}_{ir} > 0$, we cannot make the infeasibility conclusion. Thus using the definitions of blocking as under Case 1, these conditions lead to an unblocked driving variable and yet no infeasibility conclusion is possible. In order to force the algorithm to move to a successful conclusion when this occurs, we make the following modifications in the definition of blocking (the aim is to make sure that the occurrence of an unblocked driving variable indicates the infeasibility of the original system "$w - Mz = q$, $w \geqq 0$, $z \geqq 0$" through an inconsistent equation of the form (4.16)). Let $\alpha <$ minimum $\{q_i : i = 1 \text{ to } n\}$. We impose a lower bound of α on all negative variables. A negative basic variable can then block the driving variable by decreasing to its lower bound α. When this happens, the blocking negative basic variable is replaced from the basic vector by the driving variable, and made into a nonbasic variable at its lower bound α. Once any variable attains a nonnegative value its lower bound is immediately changed to zero. With this modification, each nonbasic variable either has value 0 or α. A basic solution is nondegenerate if each basic variable has value different from 0 or α in the solution. Since nonbasic variables can have nonzero values, the basic values may not be equal to the updated right hand side constant vector $\overline{q}$, so we have to maintain the basic values separately in a column called $\overline{b}$.

At any stage of this method, if $\hat{q}$, $\hat{b}$, $-\hat{m}_{ij}$ denote the updated right hand side constants vector, updated basic values vector, and the updated entries in the nonbasic columns respectively, then $\hat{b}_i = \hat{q}_i + \Sigma(\alpha \hat{m}_{ij}$: over j such that the corresponding variable is nonbasic at its lower bound α). If at this stage the driving column (the updated column of the driving variable) is $(-\hat{m}_{1s}, \ldots, -\hat{m}_{ns})^T$, and the distinguished variable is the basic variable in the rth row, it can be shown that $\hat{m}_{rs} \geqq 0$ using the facts that the PPTs of a PSD matrix are PSD, and that the principal subdeterminants of a PSD matrix are $\geqq 0$ (similar to the proof of the corresponding statement that

$\overline{\overline{m}}_{rr} > 0$ under Case 1). Compute $\theta = \text{minimum } \{(-\hat{b}_r/\hat{m}_{rs})$, if $\hat{m}_{rs} \neq 0$; $(-\hat{b}_i/\hat{m}_{is})$, for all i such that $-\hat{b}_i \geqq 0$ and $\hat{m}_{is} < 0$; $(\alpha - \hat{b}_i)/\hat{m}_{is}$, for all i such that $\hat{b}_i < 0$ and $\hat{m}_{is} < 0\}$. The blocking variable is the ith basic variable corresponding to the i that attains the minimum here. Ties for the blocking variable should be resolved using the lexico minimum ratio test in place of the usual minimum ratio test as described above. If a blocking variable exists, the pivot step replaces the blocking variable in the basic vector by the driving variable. In the new basic solution obtained after the pivot step the blocking variable that just left the basic vector is zero if it was the distinguished variable or a nonnegative basic variable, or α if it was a negative valued basic variable that decreased to its lower bound. The old driving variable which is now the new rth basic variable, has a value of θ in the basic solution. The new value of the ith basic variable is $\hat{b}_i + \theta\hat{m}_{is}$ for $i \neq r$. All other variables (nonbasics) continue to have the same value in the basic solution as before. If the distinguished variable is still basic, the procedure is continued by choosing the new driving variable to be the complement of the blocking variable that just dropped from the basic vector. As before, the procedure does not allow any nonnegative variable to become negative. It can be verified that each iteration of the method results in an increase (or lexico increase) of the sum of the distinguished variable and its complement. The major cycle terminates when the distinguished variable reaches the value zero and drops out of the basic vector, leading to a complementary basic vector.

To choose the distinguished variable at the beginning of a major cycle, we look for a basic variable, say the rth, whose value in the current basic solution, $\overline{b}_r < 0$ (even though the current updated $\overline{q}_r$ may be $\geqq 0$). However, in this case it is possible that no such basic variable exists. This happens when we reach a complementary basic vector with nonnegative values for all the basic variables in the current basic solution. If all the nonbasic variables are zero in this solution, the present complementary basic vector is feasible to the original LCP (q, M) and we terminate. On the other hand, if there are some nonbasic variables which are at their lower bound α in the current solution, check whether the current updated right hand side constants vector $\overline{q}$ is $\geqq 0$. If so, set all the nonbasic variables to zero, this changes the basic values to $\overline{q}$, and since $\overline{q} \geqq 0$, the present complementary basic vector is feasible to the original LCP (q, M) and we terminate. However, if $\overline{q} \not\geqq 0$ in such a situation, select one of the negative nonbasic variables (with value $= \alpha$ in the present basic solution) as the distinguished variable. In the first step of the ensuing major cycle, that nonbasic distinguished variable is itself the driving variable. If it is blocked, it becomes a basic variable after the first pivot step, and the major cycle continues until this distinguished variable increases to zero. However, a major cycle like this in which the nonbasic distinguished variable is the driving variable may consist of one step without any pivots if this driving variable can increase all the way from α to zero without making any nonnegative basic variable negative.

If we have a complementary basic vector in which the driving variable is unblocked, it cannot be the distinguished variable (since a distinguished driving variable must be a negative nonbasic variable which will not be increase beyond zero). So an unblocked driving variable when the present basic vector is complementary must be the complement of a negative basic variable upon which its increase has no effect. Being unblocked, the updated column of the driving variable must be $\leqq 0$, and this implies infeasibility of the original LCP as discussed earlier.

The pivot element in any almost complementary basic vector is always positive by the rules under which the method is operated. The pivot element is only negative in this method when the dropping basic variable is the distinguished variable, which signals the end of a major cycle.

Suppose the driving variable is unblocked when the present basic vector is almost complementary. When this happens, the distinguished variable must be basic. Suppose it is the rth. Its complement must also be basic. Suppose it is the pth basic variable. Let the updated column of the driving variable be $(-\hat{m}_{1s}, \dots, -\hat{m}_{ns})^T$. Since the distinguished variable is not blocking, we must have $\hat{m}_{rs} = 0$. Also we must have $-\hat{m}_{is} \leqq 0$, as otherwise some basic variable would block. It can be verified that in this case $-\hat{m}_{ps} < 0$. Pivoting with $-\hat{m}_{ps}$ as the pivot element restores complementarity and it can be verified that after this pivot step, it is possible to conlcude that the original LCP is infeasible.

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