The Balanced Transportation Problem

Katta G. Murty Lecture slides

Input Data :

SOURCES: m sources with supplies available of $a_1, \dots, a_m$.

SINKS: n sinks with demands of $b_1, \dots, b_n$.

COST MATRIX: (c_{ij}) matrix of unit shipping cost coefficients.

FORBIDDEN CELLS: Each source may be allowed to ship only to a subset of sinks; i.e., it is forbidden to ship to sinks outside this subset. Variables associated with forbidden cells are required to be 0. Make cost coeff. c_{ij} of forbidden cells (i, j) to be $+\infty$ or a very large positive number, to force variable to be 0.

Balance Condition: Total supply available = Total demand; i.e., $\Sigma_{i=1}^{m} a_i = \Sigma_{j=1}^{n} b_j$.

Decision Variables: x_{ij} = amount shipped from source i to sink j, $i = 1$ to m, $j = 1$ to n.

Problem:

$$
\begin{aligned}
\min \quad & z(x) = \sum\sum c_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1 \text{ to } m \\
& \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1 \text{ to } n \\
& x_{ij} \geq 0 \quad \text{for all } i, j.
\end{aligned}
$$

Redundancy in constraints: There is a redundant constraint, and any constraint in system can be treated as redundant one and eliminated.

We will treat constraint corresponding to $j = n$ as redundant and eliminate it. Effect is to set associated dual variable $v_n = 0$ in dual problem.

Dual Problem:

$$
\begin{aligned}
\max \quad & w(u, v) = \sum a_i u_i + \sum b_j v_j \\
\text{subject to} \quad & u_i + v_j \leq c_{ij} \quad \text{for all } i, j \\
& v_n = 0
\end{aligned}
$$

Relative cost coeff. of x_{ij} is dual slack $\bar{c}_{ij} = c_{ij} - u_i - v_j$. Various complementary pairs are $(x_{ij}, \bar{c}_{ij})$.

Basic vectors: Every basic vector has $(m + n - 1)$ basic variables.

Lower Triangular, Upper Triangular, Triangular Matrices

A triangular matrix has properties:

1. It has a row with a single nonzero entry.
2. Submatrix obtained by striking off the row and col. of a unique nonzero entry in a row again has property 1.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

A square matrix A is triangular iff it is nonsingular & both $Ax = b$, & $\pi A = c$, can be solved by **backsubstitution**.

Theorem: Every basis for the balanced transportation problem is triangular.

Theorem: The determinant of every basis for the balanced transportation problem is ± 1. As a consequence, every basic solution is integral if the RHS constants vector is integral.

In following, the cost coeffs. in basic cells in a basic vector are marked with a box. Find primal & dual basic sols. by backsubstitution.

	C_1	C_2	C_3	C_4	C_5	C_6	
R_1			$\boxed{7}$	$\boxed{10}$			213
R_2	$\boxed{25}$					$\boxed{15}$	312
R_3		$\boxed{33}$			$\boxed{9}$		111
R_4		$\boxed{11}$		$\boxed{4}$		$\boxed{8}$	110
	163	22	100	318	79	64	

How to Determine if a subset of $m+n-1$ cells is a basic set?

1. There must exist a line in the array (row or col) with a single cell from subset.

2. The remaining array after a line with a single cell from subset is crossed out must againg satisfy Property 1.

Special Routine to find Initial BFS:

General Step: Suppose some basic cells have already been selected, and some rows and some columns of the array are struck-off from further consideration. Let a_i', b_j' denote modified row and column totals.

If remaining portion of array consists of a single row only [single column only], select each remaining cell in it as a basic cell, make value of basic variable corresponding to it = modified column [row] total at this stage, terminate.

If remaining portion of array has ≥ 2 rows and ≥ 2 cols., select one of remaining cells, (r, s) say, as next basic cell.

BASIC CELL SELECTION RULES: 1: GREEDY CHOICE

RULE: Select (r, s), the one with smallest cost coeff. among ones open (i.e., not struck off) at this stage.

2. VOGEL's CHOICE RULE: In each row and col. not yet struck off, define Cost Difference (CD)

$$\text{CD} = \text{2nd smallest cost coeff.} - \text{smallest cost coeff. among open cells.}$$

Find row or col. with largest CD. Select (r, s), the open cell in it with smallest cost coeff.

NOTE: Forbidden cells may be selected as basic cells. OK. Process of cost minimization will force them out of basic vector later, if a feasible solution exists.

Fix value of basic variable $x_{rs} = \beta = \min\{a'_r, b'_s\} \geq 0$ in the BFS.

If $\beta = a'_r$, strike off row r from further consideration, replace b'_s by $b'_s - \beta$ (even if it is 0), and go to next step.

If $\beta < a'_r$, i.e., $\beta = b'_s$, replace a'_r by $a'_r - \beta$, strike off col. s from further consideration, and go to next step.

Examples:

	c_{ij}			a_i
	$j = 1$	2	3	
$i = 1$	11	8	2	800
2	7	5	4	300
b_j	400	500	200	

	c_{ij}			a_i
	$j = 1$	2	3	
$i = 1$	1	2	3	15
2	4	4	10	19
3	6	5	15	11
b_j	7	8	30	

How to check Optimality of a BES:

1. Compute associated dual basic solution.

   Dual constraint associated with primal variable x_{ij} is $u_i+v_j \leq c_{ij}$.

   Dual solution associated with a basic vector is obtained by solving the system of dual constraints associated with basic variables as a system of linear equations. So, here we need to solve:

   $$u_i + v_j = c_{ij} \quad \text{for all basic cells } (i, j)$$
   $$v_n = 0$$

   System of $m+n$ equations in $m+n$ unknowns, can be solved by backsubstitution beginning with $v_n = 0$.

   Example: Basic vector $(x_{11}, x_{12}, x_{13}, x_{22})$ for first example above.

2. Compute relative cost coeffs. of nonbasic variables.

   Relative cost coeff. of nonbasic variable x_{pq} is the dual slack associated with x_{pq}, $\bar{c}_{pq} = c_{pq} - u_p - v_q$.

3. Check Optimality.

Optimality Criterion: $\bar{c}_{ij} \geq 0$ for all nonbasic (i, j).

3.1. If optimality criterion satisfied, present solutions optimal. Primal BFS gives amounts to be shipped to various sinks from each source.

In optimum solution, if some forbidden basic variable has positive value, then original problem has no feasible solution, i.e., no solution without shipment among forbidden cells.

3.2. If optimality criterion violated, cells in $E = \{(i, j) : \bar{c}_{ij} < 0\}$ are eligible to enter basic set. Select one of them as Entering cell.

4. Find θ-loop of entering cell.

(i) A θ-loop is a subset of cells in the array marked with $+\theta$ or $-\theta$ adjustments, beginning with a $+\theta$ adjustment in the entering cell.

(ii) All cells in θ-loop other than the entering cell are basic cells.

(iii) If any row or column of array contains some cells of θ-loop, then it contains exactly 2 cells, one with a $+\theta$ adjustment, and another with a $-\theta$ adjustment.

(iv) No proper subset of a θ-loop satisfies (iii).

Cells with $+\theta$ adjustment are called Recipient cells; those with $-\theta$ adjustment are called Donor cells.

Example: In a 4×5 array, can $\{(1,2),(1,3),(2,2),(2,3),$ $(3,4),(3,5),(4,4),(4,5)\}$ form a θ-loop?

Array 6.7

	1	2	3	4	5	u_i
1	1 6	 $\boxed{18-\theta}$ 3	 $\boxed{20+\theta}$ 4	7 5	-6 5	11
2	 $\boxed{27-\theta}$ 10	7 15	6 15	11 14	 $\boxed{35+\theta}$ 16	16
3	 $\boxed{0+\theta}$ 8	-1 5	 $\boxed{27-\theta}$ 7	10 11	-5 9	14
4	-6 13	-13 θ 4	-2 16	 $\boxed{25}$ 12	 $\boxed{65-\theta}$ 25	25
v_j	-6	-8	-7	-13	0	

A trial and error method for finding the θ-loop starts by putting a $+\theta$ adjustment in entering cell. Now make a $-\theta$ adjustment among one of the basic cells in the row of entering cell. Continue making alternately $+\theta$ and $-\theta$ adjustments among basic cells until loop is complete. At any time, if stuck, backtrack and change position of previous entries and try.

5. Find dropping basic variable and new BFS.

To find new solution, fix all nonbasics other than entering variable at 0, make entering variable $= \theta$, and reevaluate values of basic variables. This is equal to:

Present BFS + θ-adjustments in θ-loop of entering cell.

Objective value of this solution is $z(\theta)$ = Present objective value + $\bar{c}_{pq}\theta$.

Actually, $\bar{c}_{pq}$ = Cost of θ-loop = sum of original cost coeffs. of recipient cells − sum of original cost coeffs. of donor cells.

Since $z(\theta)$ decreases as θ increases from 0, we should give θ maximum possible value that keeps solution ≥ 0. This is equal to:

Min ratio in this pivot step = min{Value of donor variables in present BFS }.

So, the dropping basic variable can be taken to be any donor variable which ties for minimum above.

So, pivot step consists of obtaining new BFS by substituting θ = min ratio in the new solution; replacing dropping basic variable by entering variable in the basic set.

With the new BFS and basic set, return to next iteration.

Example:

	c_{ij}			a_i
	$j=1$	2	3	
$i=1$	9	6	8	6
2	10	5	12	11
3	11	13	20	4
b_j	3	4	14	

How to find θ-loops efficiently? Tree Labels

A basis has no cycles, it forms a spanning tree. A basic vector + one other cell has a unique cycle or θ-loop.

H. I. Scoins, "The compact representation of a rooted tree and the transportation problem", 1964 Int. Symp. on MP, presented tree label method to find these loops efficiently.

	C_1	C_2	C_3	C_4	C_5	C_6	
R_1			$\boxed{7}$	$\boxed{10}$			213
R_2	$\boxed{25}$					$\boxed{15}$	312
R_3		$\boxed{33}$			$\boxed{9}$		111
R_4		$\boxed{11}$		$\boxed{4}$		$\boxed{8}$	110
	163	22	100	318	79	64	

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	
R_1					$\boxed{8}$	$\boxed{11}$			9
R_2		$\boxed{0}$							3
R_3	$\boxed{20}$			$\boxed{13}$			$\boxed{8}$		22
R_4							$\boxed{9}$		17
R_5		$\boxed{7}$	$\boxed{8}$		$\boxed{10}$		$\boxed{4}$	$\boxed{11}$	43
	1	11	19	12	6	5	34	6	

If Balance Condition Does Not Hold ...

We either have excess supply or excess demand.

Total supply > Total demand

Let Δ = Total supply − Total demand.

A total of Δ units will be left unused at all sources put together.

Introduce a dummy sink with demand = Δ. This sink represents leaving material unused at source itself. All cost coeffs. to dummy sink are 0.

Now problem balanced. Solve it. In optimal sol. shipments to dummy sink represent material unused at source.

Total Demand > Total Supply.

So, shortage = Δ = Total demand − total supply.

To meet demand, a new source of supply for Δ units must be found. If none available, open a dummy source with supply of Δ, representing unfulfilled demand at sinks. Make cost of shipping from dummy source to every sink 0.

Now problem balanced. Solve it. In optimal sol. shipments from dummy source represent unfulfilled demand at sinks.

Marginal analysis, Changes in Demand - Supply data:

Let $\bar{x}, (\bar{u}, \bar{v})$ be optimum pair. Suppose demand at sink j is changing from present b_j to $b_j + \delta$ where $\delta \geq 0$. What is the best source for increasing supply by δ to meet this additional demand, if our aim is to keep total transportation cost as small as possible?

From marginal analysis, we know that the best source is the source that has smallest $\bar{u}_i$. Suppose $\bar{u}_p = \min\{\bar{u}_i\}$.

Then increase supply at source p from present a_p to $a_p + \delta$. The change in total transportation cost will be $\delta(\bar{u}_p + \bar{v}_j)$ as long as δ is in its optimality range.

To find new optimum solution as a function of δ, find δ-path from row p to column j in array. This δ-path consists alternately of entries of $+\delta$ and $-\delta$ among basic cells only, beginning with a $+\delta$ adjustment in row p, and ending with a $+\delta$ adjustment in column j.

The optimality range of δ is the interval of δ that keeps this solution ≥ 0.

Example:

	1	2	3	4	a_i	u_i
1	$\boxed{10-\delta}$ 25	36	$\boxed{15}$ 20	$\boxed{20+\delta}$ 10	45	10
2	47	$\boxed{40+\delta}$ 40	30	$\boxed{30-\delta}$ 20	70	20
3	$\boxed{50+\delta}$ 20	33	15	14	$50+\delta$	5
b_j	60	$40+\delta$	15	50		
v_j	15	20	10	0		