Marginal and Sensitivity Analyses

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Consider LP in *standard form*: min $z = cx$, subject to $Ax = b, x \geq 0$ where $A_{m \times n}$ and rank m.

Theorem: If this LP has an optimum nondegenerate BFS, then its dual opt. sol. is unique, and it is the *marginal value vector* for this LP.

Theorem: If this LP has an optimum sol., but no optimal nondegenerate BFS, then the dual opt. sol. may not be unique. In this case, marginal value vector may not exist, but *positive and negative marginal values* exist for each b_i. They are:

$$\text{Positive MV wrt } b_i \quad = \quad \text{Max}\{\pi_i : \text{over dual opt. sols. } \pi\}$$

$$\text{Negative MV wrt } b_i \quad = \quad \text{Min}\{\pi_i : \text{over dual opt. sols. } \pi\}$$

Let $f(b)$ denote the optimum objective value function as a function of the RHS constants vector b. $f(b)$ defined only over $b \in \text{Pos}(A)$.

1. If $f(b) = -\infty$ for some $b \in \mathrm{Pos}(A)$, then it is $-\infty$ for all $b \in \mathrm{Pos}(A)$.

2. Positive Homogeneity: If $f(b)$ finite for some b, then $f(0) = 0$ and $f(\lambda b) = \lambda f(b)$ for all $\lambda \geq 0$.

3. Convexity: $f(b)$ is a piecewise linear convex function defined over $\mathrm{Pos}(A)$.

4. Subgadient Property Let π^1 be a dual opt. sol. when $b = b^1$. Then

$$f(b) \geq \pi^1 b \quad \text{for all } b \in \mathrm{Pos}(A)$$

i.e., π^1 is a subgradient of $f(b)$ at b^1.

Sensitivity Analysis

Also called *Post-optimality analysis*. Deals with efficient techniques for finding new opt. when small changes occur in data.

Consider LP in *standard form*: min $z = cx$, subject to $Ax = b, x \geq 0$ where $A_{m\times n}$ and rank m.

Example:

Original tableau

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b
1	2	0	1	0	-6	0	11
0	1	1	3	-2	-1	0	6
1	2	1	3	-1	-5	0	13
3	2	-3	-6	10	-5	1	0

$x_j \geq 0$ for all j, min z

Optimum Inverse tableau

Basic var.	Inverse tableau				Basic values
x_1	-1	-2	2	0	3
x_2	1	1	-1	0	4
x_3	-1	0	1	0	2
$-z$	-2	4	-1	1	-11

We assume that we have an opt. inverse tableau for the problem. Let x_B be the present opt. basic vector, and $\bar{x}, \bar{\pi}$ the primal and dual opt. sols.

Cost Coefficient Ranging

This finds the *optimality interval* for each original cost coefficient.

OPTIMALITY INTERVAL OF A DATA ELEMENT = Set of all values of that data element, as it varies in the tableau, but all other data remains fixed at current values, for which the present opt. basic vector (or present opt. sol.) remain optimal.

1. Nonbasic Cost Coefficient Ranging

Let x_j be a present nonbasic variable. c_j = cost coeff. of x_j which may change from its present value, while all other data remains fixed at current values.

With this change, the only thing that will change is the rel. cost coeff. of x_j, $\bar{c}_j = c_j - \pi A_{.j}$.

$\bar{c}_j$ remains ≥ 0 as long as $c_j \geq \bar{\pi} A_{.j}$. So, optimality interval for c_j is $[\bar{\pi} A_{.j}, \infty]$.

If c_j becomes $< \bar{\pi} A_{.j}$, enter x_j into x_B, and continue simplex iterations until termination again.

Example: Find range for c_5. Find new opt. sol. if c_5 changes from present 10 to 6.

Cost Ranging a Basic Cost Coeff.

Let x_p be a basic variable in the present opt. basic vector x_B. If its cost coeff. c_p changes, the dual sol. changes. So, to compute opt. interval for c_p do the following:

- In c_B replace cost coeff. of basic var. x_p by parameter c_p, and denote it by $c_B(c_p)$.

$\pi(c_p) =$ dual basic sol. as a function of c_p $= c_B(c_p)B^{-1}$

Each component of $\pi(c_p)$ is an affine function of c_p, i.e., has the form $\pi_i^0 + c_p\pi_i^1$ where π_i^0, π_i^1 are constants.

- Now compute each nonbasic relative cost coeff. $\bar{c}_j$ as a function of c_p, it is given by:

$$\bar{c}_j(c_p) = c_j - \pi(c_p)A_{.j}$$

  Again each of these is an affine function of c_p. Express the cond. that all these rel. cost coeffs. must be ≥ 0. This yields the opt. int. for c_p.

- To get new opt. sol. when c_p changes to a value outside its opt. interval, compute all nonbasic $\bar{c}_j(c_p)$, and if some of them are < 0, select one of the corresponding variables as the entering variable, and continue the application of the simplex algo.

Example: Find opt. range for c_1 and new opt. if c_1 changes to 5.

Ranging RHS Constants

Optimality interval for an RHS constant b_i = set of all values of b_i for which present opt. basic vector x_B remains opt., as b_i varies while all other data remains fixed at current values.

To find opt int. for a b_i, replace its present value in RHS constants vector b by parameter b_i, and denote it by $b(b_i)$.

The basic values vector as a function of b_i is $B^{-1}b(b_i)$.

Each component of this vector is an affine function of b_i. Express the cond. that each of them must be ≥ 0, this yields the opt. int. for b_i.

As b_i varies in its opt. int., the dual opt. sol. remains unchanged, but the primal opt. sol. is given by:

$$
\begin{aligned}
\text{Nonbasic variables} &= 0 \\
\text{Basic vector} &= B^{-1}b(b_i) \\
\text{opt. obj value} &= \bar{\pi}b(b_i)
\end{aligned}
$$

If new value of b_i is outside its opt. int., to find new opt. sol.

use dual simplex iterations.

Example: Find opt. range for b_1, and new opt. sol. when b_1 changes to 15.

Ranging Input-Output Coeffs. in a Nonbasic col.

Let x_j be a present nonbasic variable, and a_{ij} an input-output coeff. in its original column $A_{.j}$.

To find opt. range for a_{ij}, replace its present value in the column $A_{.j}$ by the parameter a_{ij}, and call it $A_{.j}(a_{ij})$.

Then express the condition that the relative cost coeff. of x_j, $\bar{c}_j(a_{ij}) = c_j - \bar{\pi}A_{.j}(a_{ij}) \geq 0$.

This yields the opt. int. for a_{ij}.

If a_{ij} changes to a value outside its opt. int., to get the new opt. sol. enter x_j into the present basic vector x_B, and continus the application of the simplex algo.

To Introduce a New Nonnegative Variable

Find the rel. cost coeff. of new variable wrt present opt. basic vector x_B.

If this is ≥ 0, extend $\bar{x}$ by including new variable at 0-value; this is the new opt. sol.

If rel. cost coeff. of new var. < 0, bring it into x_B, and continue the application of simplex algo.

Example: Include $x_7 \geq 0$ with col. vector $(1, 2, 3, \vdots - 7)^T$.

What happens if cost coeff. of new variable is -10 instead of -7?

Introducing a new Inequality Constraint

Let new constraint be $\quad A_{(m+1)\,.}\, x \leq b_{m+1}$.

If present opt. sol. $\bar{x}$ satisfies it, it remains opt., terminate.

If $\bar{x}$ violates it, let x_{n+1} be the nonnegative slack var. corresponding to new inequality. Construct inverse tableau corresponding to basic vector (x_B, x_{n+1}). This basic vector is primal infeasible to augmented problem, but dual feasible. Apply dual simplex iterations to get new opt.

$$\begin{pmatrix} B & 0 \\ a & 1 \end{pmatrix}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ -aB^{-1} & 1 \end{pmatrix}$$

$$\begin{pmatrix} B & 0 \\ a & -1 \end{pmatrix}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ aB^{-1} & -1 \end{pmatrix}$$