8.1

Marginal and Sensitivity Analyses

Consider LP in *standard form*: \( \min z = cx \), subject to \( Ax = b, x \geq 0 \) where \( A_{m \times n} \) and rank \( m \).

**Theorem:** If this LP has an optimum nondegenerate BFS, then its dual opt. sol. is unique, and it is the *marginal value vector* for this LP.

**Theorem:** If this LP has an optimum sol., but no optimal nondegenerate BFS, then the dual opt. sol. may not be unique. In this case, marginal value vector may not exist, but *positive and negative marginal values* exist for each \( b_i \). They are:

\[
\begin{align*}
\text{Positive MV wrt } b_i &= \max \{ \pi_i : \text{over dual opt. sols. } \pi \} \\
\text{Negative MV wrt } b_i &= \min \{ \pi_i : \text{over dual opt. sols. } \pi \}
\end{align*}
\]

Let \( f(b) \) denote the optimum objective value function as a function of the RHS constants vector \( b \). \( f(b) \) defined only over \( b \in \text{Pos}(A) \).
1. If \( f(b) = -\infty \) for some \( b \in \text{Pos}(A) \), then it is \(-\infty\) for all \( b \in \text{Pos}(A) \).

2. **Positive Homogeneity:** If \( f(b) \) finite for some \( b \), then \( f(0) = 0 \) and \( f(\lambda b) = \lambda f(b) \) for all \( \lambda \geq 0 \).

3. **Convexity:** \( f(b) \) is a piecewise linear convex function defined over \( \text{Pos}(A) \).

4. **Subgradient Property** Let \( \pi^1 \) be a dual opt. sol. when \( b = b^1 \). Then

\[
f(b) \geq \pi^1 b \quad \text{for all } b \in \text{Pos}(A)
\]

i.e., \( \pi^1 \) is a subgradient of \( f(b) \) at \( b^1 \).
8.2

Sensitivity Analysis

Also called *Post-optimality analysis*. Deals with efficient tech-
niques for finding new opt. when small changes occur in data.

Consider LP in *standard form*: $\min z = cx$, subject to $Ax = b, x \geq 0$ where $A_{m \times n}$ and rank $m$.

Example:

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$-z$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-6</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>-1</td>
<td>-5</td>
<td>0</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>-3</td>
<td>-6</td>
<td>10</td>
<td>-5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$x_j \geq 0$ for all $j$, $\min z$
We assume that we have an opt. inverse tableau for the problem. Let $x_B$ be the present opt. basic vector, and $\bar{x}, \bar{\pi}$ the primal and dual opt. sols.

Cost Coefficient Ranging

This finds the *optimality interval* for each original cost coefficient.

**OPTIMALITY INTERVAL OF A DATA ELEMENT** = Set of all values of that data element, as it varies in the tableau, but all other data remains fixed at current values, for which the present opt. basic vector (or present opt. sol.) remain optimal.
1. Nonbasic Cost Coefficient Ranging

Let $x_j$ be a present nonbasic variable. $c_j = \text{cost coeff. of } x_j$ which may change from its present value, while all other data remains fixed at current values.

With this change, the only thing that will change is the rel. cost coeff. of $x_j$, $\bar{c}_j = c_j - \pi A_{.j}$.

$\bar{c}_j$ remains $\geq 0$ as long as $c_j \geq \pi A_{.j}$. So, optimality interval for $c_j$ is $[\pi A_{.j}, \infty]$.

If $c_j$ becomes $< \pi A_{.j}$, enter $x_j$ into $x_B$, and continue simplex iterations until termination again.

Example: Find range for $c_5$. Find new opt. sol. if $c_5$ changes from present 10 to 6.

Cost Ranging a Basic Cost Coeff.

Let $x_p$ be a basic variable in the present opt. basic vector $x_B$. If its cost coeff. $c_p$ changes, the dual sol. changes. So, to compute opt. interval for $c_p$ do the following:

- In $c_B$ replace cost coeff. of basic var. $x_p$ by parameter $c_p$. 

163
and denote it by $c_B(c_p)$.

$$\pi(c_p) = \text{dual basic sol. as a function of } c_p = c_B(c_p)B^{-1}$$

Each component of $\pi(c_p)$ is an affine function of $c_p$, i.e., has the form $\pi_i^0 + c_p\pi_i^1$ where $\pi_i^0, \pi_i^1$ are constants.

- Now compute each nonbasic relative cost coeff. $\bar{c}_j$ as a function of $c_p$, it is given by:

$$\bar{c}_j(c_p) = c_j - \pi(c_p)A_{,j}$$

Again each of these is an affine function of $c_p$. Express the cond. that all these rel. cost coeffs. must be $\geq 0$. This yields the opt. int. for $c_p$.

- To get new opt. sol. when $c_p$ changes to a value outside its opt. interval, compute all nonbasic $\bar{c}_j(c_p)$, and if some of them are $< 0$, select one of the corresponding variables as the entering variable, and continue the application of the simplex algo.

Example: Find opt. range for $c_1$ and new opt. if $c_1$ changes to 5.
Ranging RHS Constants

Optimality interval for an RHS constant $b_i = \text{set of all values of } b_i \text{ for which present opt. basic vector } x_B \text{ remains opt.},$ as $b_i$ varies while all other data remains fixed at current values.

To find opt int. for a $b_i$, replace its present value in RHS constants vector $b$ by parameter $b_i$, and denote it by $b(b_i)$.

The basic values vector as a function of $b_i$ is $B^{-1}b(b_i)$.

Each component of this vector is an affine function of $b_i$. Express the cond. that each of them must be $\geq 0$, this yields the opt. int. for $b_i$.

As $b_i$ varies in its opt. int., the dual opt. sol. remains unchanged, but the primal opt. sol. is given by:

Nonbasic variables $= 0$

Basic vector $= B^{-1}b(b_i)$

opt. obj value $= \pi b(b_i)$

If new value of $b_i$ is outside its opt. int., to find new opt. sol.
use dual simplex iterations.

Example: Find opt. range for $b_1$, and new opt. sol. when $b_1$ changes to 15.

Ranging Input-Output Coeffs. in a Nonbasic col.

Let $x_j$ be a present nonbasic variable, and $a_{ij}$ an input-output coeff. in its original column $A_{.j}$.

To find opt. range for $a_{ij}$, replace its present value in the column $A_{.j}$ by the parameter $a_{ij}$, and call it $A_{.j}(a_{ij})$.

Then express the condition that the relative cost coeff. of $x_j$, $\bar{c}_j(a_{ij}) = c_j - \bar{\pi}A_{.j}(a_{ij}) \geq 0$.

This yields the opt. int. for $a_{ij}$.

If $a_{ij}$ changes to a value outside its opt. int., to get the new opt. sol. enter $x_j$ into the present basic vector $x_B$, and continue the application of the simplex algo.

To Introduce a New Nonnegative Variable

Find the rel. cost coeff. of new variable wrt present opt. basic vector $x_B$. 
If this is $\geq 0$, extend $\bar{x}$ by including new variable at 0-value; this is the new opt. sol.

If rel. cost coeff. of new var. $< 0$, bring it into $x_B$, and continue the application of simplex algo.

**Example:** Include $x_7 \geq 0$ with col. vector $(1, 2, 3, \ldots, -7)^T$.

What happens if cost coeff. of new variable is $-10$ instead of $-7$?

**Introducing a new Inequality Constraint**

Let new constraint be $A_{(m+1)} \cdot x \leq b_{m+1}$.

If present opt. sol. $\bar{x}$ satisfies it, it remains opt., terminate.

If $\bar{x}$ violates it, let $x_{n+1}$ be the nonnegative slack var. corresponding to new inequality. Construct inverse tableau corresponding to basic vector $(x_B, x_{n+1})$. This basic vector is primal infeasible to augmented problem, but dual feasible. Apply dual simplex iterations to get new opt.

$$\left( \begin{array}{cc} B & 0 \\ a & 1 \end{array} \right)^{-1} = \left( \begin{array}{cc} B^{-1} & 0 \\ -aB^{-1} & 1 \end{array} \right)$$
$$\begin{pmatrix} B & 0 \\ a & -1 \end{pmatrix}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ aB^{-1} & -1 \end{pmatrix}$$