Dual Simplex Algorithm

Katta G. Murty, IOE 510, LP, U. Of Michigan, Ann Arbor, Winter 1997.

Example: Consider the following LPs:

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b
1	0	0	-1	-2	2	0	-6
0	1	0	1	-1	-2	0	-2
0	0	1	1	1	-1	0	4
0	0	0	3	8	10	1	-100

$x_j \geq 0 \quad \forall j$, min z

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1	0	0	-1	-2	2	0	-6
0	1	0	1	2	0	0	-2
0	0	1	1	-1	1	0	4
0	0	0	3	8	10	1	-100

$x_j \geq 0 \quad \forall j$, min z

Consider LP in *standard form*: min $z = cx$, subject to $Ax = b, x \geq 0$ where $A_{m \times n}$ and rank m.

Dual simplex algorithm is just the opposite of the primal simplex algo. Starting with a dual feasible basis (i.e., one in which $\bar{c}_j \geq 0$ for all j) it tries to attain primal feasibility while maintaining dual feasibility throughout. *All operations are carried out on the primal simplex tableaus themselves.*

The Algorithm

INPUTS NEEDED: LP in standard form, a dual feasible basic vector.

INITIAL SETUP: Compute the inverse tableau corresponding to the initial dual feasible basic vector.

GENERAL ITERATION: Let x_B be present dual feasible basic vector, and let inverse tableau be:

Inverse tableau wrt x_B

Basic var.	Inverse tableau		Basic values
x_B	B^{-1}	0	$\bar{b}_1$ $\vdots$ $\bar{b}_i$ $\vdots$ $\bar{b}_m$
$-z$	$-\bar{\pi}$	1	$-z^0$

All nonbasics $= 0$

Since x_B dual feasible, the nonbasic relative costs $\bar{c}_j$ are all ≥ 0.

OPTIMALITY CRITERION: If $\bar{b}_i \geq 0$ for all $i = 1$ to m, x_B opt. basic vector, and present BFS an opt. sol., terminate.

PIVOT ROW SELECTION: $F = \{i : \bar{b}_i < 0\}$. For each $i \in F$, roe i eligible to be choosen as pivot row. Select one of these rows, say $r \in F$ as *pivot row.* So, the present rth basic variable is the *dropping basic variable* in this step.

PIVOT COL. SELECTION: THE DUAL SIMPLEX MIN RATIO TEST: The entering variable (and hence the pivot col. which will be its updated col.) is selected to make sure that the next basic vector is also dual feasible. Involves following work:

1. Compute pivot row = updated rth row =

$$(\bar{A}_{r.} \vdots \bar{b}_r) = (B^{-1})_{r.}(A \vdots b) = (\bar{a}_{r1}, \ldots, \bar{a}_{rn} \vdots \bar{b}_r)$$

2. *PRIMAL INFEASIBILITY CRITERION:* If pivot row = $(\bar{a}_{r1}, \ldots, \bar{a}_{rn}) \geq 0$, the equation corresponding to it is:

$$\bar{a}_{r1}x_1 + \ldots + \bar{a}_{rn}x_n = \bar{b}_r$$

   with RHS constant $\bar{b}_r < 0$, and coefficients of all variables ≥ 0. So, it cannot have a nonnegative solution, i.e., primal system infeasible. Terminate algo. and go to infeasibility analysis.

3. *DUAL SIMPLEX MIN RATIO TEST:* If $(\bar{a}_{r1}, \ldots, \bar{a}_{rn}) \not\geq 0$, let $\bar{c}_j$ denote rel. cost coeffs. Find

$$\delta = \min\{-\bar{c}_j/\bar{a}_{rj} : \quad x_j \text{ nonbasic, and } \quad \bar{a}_{rj} < 0\}$$

and let $j = s$ be any of the subscripts that attains min above. Choose x_s as *entering variable*.

4. Perform pivot step and go to next iteration with new basic vector.

Examples:

Original tableau

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b
1	0	0	4	-5	7	0	8
0	1	0	-2	4	-2	0	-2
0	0	1	1	-3	2	0	2
0	0	0	1	3	2	1	0

$x_j \geq 0$ for all j, min z

Original tableau

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b
1	1	1	1	0	0	0	1
-1	-2	-1	0	1	0	0	-8
1	3	2	0	0	1	0	5
8	24	15	0	0	0	1	0

$x_j \geq 0$ for all j, min z

Properties.

1. **Each Sol. Opt. in a larger set:** Consider the problem in standard form: min $z = cx$, subject to $Ax = b, x \geq 0$. Let K denote the set of feasible solutions for this original problem.

   Let x_B be a dual feasible but primal infeasible basic vector associated with basic sol. $\bar{x}$. Let $J = \{j : \bar{x}_j < 0\}$.

   $\bar{x}$ violates the primal constraints $x_j \geq 0$ for $j \in J$. and *each of these variables x_j for $j \in J$ is a basic variable in* x_B. Let $\tilde{K}$ denote the set of feasible sols. of *relaxed problem* obtained by relaxing the constraints $x_j \geq 0$ for each $j \in J$ violated by $\bar{x}$, from the original problem. So, $\tilde{K} \supset K$ and $\bar{x} \in \tilde{K}$. Since x_B is dual feasible, we verify that $\bar{x}$ is in fact a minmum cost solution in $\tilde{K}$, i.e., an optimum sol. for the relaxed problem mentioned above.

   So, every basic sol. in the dual simplex algo., is an opt. sol. for a relaxed problem, with the set of feasible sols. larger than

K. So obj. value of current basic sol. in dual simplex algo. is always a lower bound for min obj. value in original problem.

2. The dropping variable is always one the bound restriction on which is violated by present sol. In next basic vec. this variable is nonbasic, so the bound restriction for it holds as an equation in the next basic sol. This sol. also satisfies the bound restriction on entering variable, as it has positive value in it.

3. As before, pivot step is:

nondegenerate if minratio is > 0

degenerate if min ratio is 0.

In a nondegenerate pivot step, the obj. value strictly increases. In a degenerate pivot step, the obj. value and dual sol. do not change, but primal sol. changes strictly in every pivot step.

So, in dual simplex algo. opt. obj. value is approached from below, in a monotonic increasing manner.

Usefulness of Dual Simplex Algorithm

Not used to solve new LPs, because the dual simplex min ratio test needs $O(n)$ comparisions in every pivot step (primal simplex min ratio test needs only $O(m)$ comparisons in each step, and in most real world models $n >> m$).

However, dual simplex algo. very useful in sensitivity analysis. After problem solved, if changes occur in RHS constants vector, dual simplex iterations are used to get new opt.

Dual Simplex Method

If an initial dual feasible basis not available, an artificial dual feasible basis can be constructed by getting an arbitrary basis, and then adding one artificial constraint.

Though mathematically well specified, this method not used much in practice.