Duality Theory, Optimality Conditions

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We only consider single objective LPs here. Concept of duality not defined for multiobjective LPs.

Every LP has another LP called its *dual*, which shares the same data, and is derived through *rational economic arguments*. In this context the original LP called the *primal LP*.

Variables in the dual problem are different from those in the primal; each dual variable is associated with a primal constraint, it is the *marginal value* or *Lagrange multiplier* corresponding to that constraint.

Example: Primal – Fertilizer Maker's Problem.

FERTILIZER MAKER has daily supply of:

1500 tons of RM 1

1200 tons of RM2

500 tons of RM 3

She wants to use these supplies to maximize net profit.

DETERGENT MAKER wants to buy *all of fertilizer maker's* *supplies* at cheapest price for his detergent process. Suppose he offers:

$$\$\ \pi_1/\text{ton for RM1}$$

$$\$\ \pi_2/\text{ton for RM2}$$

$$\$\ \pi_3/\text{ton for RM3}$$

These prices are the variables in his problem. Total payment comes to $1500\pi_1 + 1200\pi_2 + 500\pi_3$ which he wants to minimize.

FERTILIZER MAKER: won't sell supplies unless detergent maker's prices are competitive with each of hi-ph, lo-ph processes.

Hi-ph process converts a packet of {*2 tons RM1, 1 ton RM2, and 1 ton RM3*} into \$15 profit. In terms of detergent maker's prices, the same packet yields $\$(2\pi_1 + \pi_2 + \pi_3)$. So, she demands $2\pi_1 + \pi_2 + \pi_3 \geq 15$ for signing deal.

Similarly, by analyzing lo-ph process, she demands $\pi_1 + \pi_2 \geq 10$ to sign deal.

From these economic arguments, we see that detergent maker's problem is:

Dual						Associated primal activity
Minimize	$1500\pi_1$	$+1200\pi_2$	$+500\pi_3$			
subject to	$2\pi_1$	$+\pi_2$	$+\pi_3$	$\geq$	15	Hi-ph
	π_1	$+\pi_2$		$\geq$	10	lo-ph
	$\pi_1,$	$\pi_2,$	$\pi_3,$	$\geq$	0	

From arguments, we see that if dual problem has unique opt. sol., it is the marginal value vector for primal.

Procedure for Writing Dual of a General LP

1. Arrange all the constraints in the form of a detached coefficient tableau. Here constraint means:

- any condition involving 2 or more variables
- any nonzero lower bound or upper bound condition on a single variable.

Hence, only conditions not counted as constraints for this purpose are nonnegativity or nonpositivity restrictions on individual variables.

Suppose $x_j, j = 1$ to n are primal variables, and $\mathcal{A}_j$ is the column of x_j in this tableau. Let c_j be the coeff. of x_j in the primal obj. function. Let b be the RHS constants vector in this tableau.

Any primal variable for which a lower and an upper bound restriction are included among the constraints above, is treated as an unrestricted variable.

2. Define *right type of inequality* to mean;

- $\geq$ inequality in a minimization problem
- $\leq$ inequality in a maximization problem.

Otherwise inequality constraint called *wrong type*.

3. Associate a separate dual variable for each primal constraint above. Let π denote row vector of dual variables in proper order.

4. In dual problem,

- dual variables associated with primal equality constraints are unrestricted in sign.
- dual variables associated with right (wrong) type of primal inequality constraints are nonnegative (nonpositive) variables.

5. The dual objective function is πb. If primal ia a minimization problem, dual is a maximization problem, and vice versa.

6. There is one dual constraint for each primal variable. The one associated with x_j is:

- $\pi\mathcal{A}_j = c_j$ if x_j is an unrestricted primal variable
- $\pi\mathcal{A}_j \odot c_j$ if x_j is a sign restricted primal variable

where, $\odot$ denotes the right (wrong) type inequality for dual problem, if x_j is a nonnegative (nonpositive) variable.

5.7

Complementary Pairs In A Primal, Dual Pair

Let $(P), (D)$ be a primal, dual pair of LPs.

If x_j is a sign restricted variable in (P), there is an inequality constraint in (D) associated with it; let s_j denote the nonnegative dual slack variable corresponding to it. Then $(\oplus x_j, s_j)$ is a complementary pair in $(P), (D)$; where $\oplus x_j$ is x_j $(-x_j)$ if it is a nonnegative (nonpositive) variable in (P).

Similarly, if π_i is a sign restricted dual variable, it is associated with a primal inequality constraint. $\pm\pi_i$ whichever is nonnegative, and the nonnegative primal slack variable corresponding to it, also form a complementary pair in $(P), (D)$.

Examples

Min $z = 3x_1 + 11x_2 - 15x_3 + 10x_4 + 4x_5 + 57x_6$

subject to $x_1 + 2x_2 + 3x_3 - 2x_4 + x_5 + 16x_6 = 17$

$$x_2 - 4x_3 + x_4 + x_5 + x_6 = 2$$

$$x_3 - 2x_4 + x_5 = 1$$

$$x_j \geq 0 \text{ for all } j$$

$$\begin{aligned} \text{Minimize} \quad & z = cx \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \text{Minimize} \quad & z = 6x_1 + 7x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \geq -10 \\ & 3x_2 + 7x_3 \geq 13 \\ & x \geq 0 \end{aligned}$$

$$
\begin{aligned}
\text{Minimize} \quad & z = cx \\
\text{subject to} \quad & Ax \ \geq \ b \\
& x \ \geq \ 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Minimize} \quad & z = cx \\
\text{subject to} \quad & Ax \ \geq \ b \\
& \ell \leq x \ \leq \ u
\end{aligned}
$$

Duality Theory

1. Farkas' Lemma: Theorem of Alternatives for Linear Constraints Including Inequalities: For any $A_{m \times n}, b \in R^m$; either system (1) has a feasible solution x, or system (2) has a feasible solution π, but not both.

$$\left.\begin{array}{rcl} Ax & = & b \\ x & \geq & 0 \end{array}\right\} \qquad (1)$$

$$\left.\begin{array}{rcl} \pi A & \leq & 0 \\ \pi b & > & 0 \end{array}\right\} \qquad (2)$$

This is a special case of the *separating hyperplane theorem for two disjoint convex sets*. The main duality theorem of LP can be proved as a corollary of Farkas' lemma and vice versa.

2. Dual of the dual is the primal.

3. Duals of equivalent LPs are equivalent.

4. Both the primal and dual problems could be infeasible.

EXAMPLE:

$$
\begin{aligned}
\text{Minimize} \quad z = 2x_1 - 4x_2 & \\
\text{subject to} \quad x_1 - x_2 &= 1 \\
-x_1 + x_2 &= 2 \\
x &\geq 0
\end{aligned}
$$

5. Weak Duality Theorem: In a primal, dual pair; if primal is the minimization problem with objective function $z(x)$, and dual is the maximization problem with objective function $v(\pi)$; then, for all primal feasible solutions x and dual feasible solutions π,

$$z(x) \geq v(\pi)$$

Corollaries:

(i) The primal objective value at any primal feasible solution is an upper bound for the maximum objective value in the dual.

(ii) The dual objective value at any dual feasible solution is a lower bound for the minimum objective value in the primal.

(iii) If primal is feasible, and $z \longrightarrow -\infty$ in it, dual must be infeasible.

(iv) If dual is feasible, and $v \longrightarrow \infty$ in it, primal must be infeasible.

(v) SUFFICIENT OPT. CRITERION FOR PRIMAL, DUAL PAIR: If $\bar{x}$ is primal feasible, $\bar{\pi}$ is dual feasible, and $z(\bar{x}) = v(\bar{\pi})$, then $\bar{x}$ is opt. for primal, and $\bar{\pi}$ is opt. for dual.

EXAMPLE : Fertilizer problem: $\bar{x} = (300, 900)^T$, $\bar{\pi} = (5, 5, 0)$.

EXAMPLE: A diet problem. $\bar{x} = (0, 0, 0, 0, 5, 2)^T$, $\bar{\pi} = (3, 8)$.

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	2	2	1	2	≥ 9
0	1	3	1	3	2	≥ 19
35	30	60	50	27	22	$= z$ min.

$$x_j \geq 0 \text{ for all } j$$

6. Duality Theorem of LP: In a primal, dual pair of LPs, if one has an opt. sol., the other does also, and the two optimum objective values are equal.

7. In a primal, dual pair of LPs, if one of the problems is feasible, and has the objective unbounded, then the other problem is infeasible.

8. COMPLEMENTARY SLACKNESS THEOREM: In a primal, dual pair of LPs, let $\bar{x}, \bar{\pi}$ be primal and dual feasible respectively. They are optimal to the respective problems iff at least one quantity in every complementary pair is 0 in them.

C.S. THEOREM FOR STANDARD FORM: $A_{m\times n}$.

Primal: min $z = cx$, subject to $Ax = b, x \geq 0$.

Dual : max $v = \pi b$, subject to $\pi A \leq c$

C. pairs are: $(x_j, \bar{c}_j = c_j - \pi A_{.j})$, $\quad j = 1$ to n.

C.S. THEOREM FOR SYMMETRIC FORM: $A_{m \times n}$.

Primal: min $z = cx$, subject to $Ax \geq b, x \geq 0$.

Dual : max $v = \pi b$, subject to $\pi A \leq c, \pi \geq 0$.

C. pairs are: $(x_j, \bar{c}_j = c_j - \pi A_{.j})$, $\quad j = 1$ to n; and $(\pi, A_{i.}x - b_i)$, $\quad i = 1$ to m.

9. Nec. & suff. Opt. Conds. for LP: A feasible sol. $\bar{x}$ to an LP (the primal), is optimal iff there exists a dual feasible solution $\bar{\pi}$ which satisfies the C. S. Conds. with it.

Can also be proved from Farkas' lemma.

Example: Consider the feasible solution $\bar{x} = (6, 0, -1, 0, 2)^T$ for the LP

$$\begin{array}{lrcl}
\min & z = -3x_1 + x_2 + 3x_3 + 5x_5 & & \\
\text{s. to} & x_1 + x_2 - x_3 + 2x_4 - x_5 & \geq & 5 \\
& -2x_1 + 2x_3 - x_4 + 3x_5 & \geq & -8 \\
& x_1 & \geq & 6 \\
& -3x_2 + 3x_4 & \geq & -5 \\
& 5x_3 - x_4 + 7x_5 & \geq & 7
\end{array}$$

9. Strict Complementary Slackness Theorem: Suppose a primal, dual pair of LPs have optimum solutions. Then there exist optimum solutions to the two problems in which exactly one variable in every complementary pair is zero and the other is positive.

LP in Standard Form, Primal and Dual Feasibility of Basic Vectors Consider the LP in standard form, for which the original tableau given below.

Original Tableau

Basic	x_1	$\dots$	x_m	x_{m+1}	$\dots$	x_n	$-z$	
x_1	a_{11}	$\dots$	a_{1m}	$a_{1,m+1}$	$\dots$	a_{1n}	0	b_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_m	a_{m1}	$\dots$	a_{mm}	$a_{m,m+1}$	$\dots$	a_{mn}	0	b_m
$-z$	c_1	$\dots$	c_m	c_{m+1}	$\dots$	c_n	1	0

A basic vector, $x_B = (x_1, \dots, x_m)$ say, and the associated basis B, are said to be:

Primal feasible	If the primal basic solution wrt x_B is primal feasible, i.e., ≥ 0, i.e., if $B^{-1}b \geq 0$.
Dual feasible	Dual basic sol. obtained by solving system of dual constraints corresponding to basic variables in x_B as a system of eqs., i.e., $\pi B = c_B$. So, dual basic sol. is $\pi = c_B B^{-1}$. This definition guarantees that the primal & dual basic sols. wrt a basic vector always satisfy C.S. conds. x_B said to be dual feasible if the dual basic sol. satisfies other dual constraints; i.e., if dual slacks $\bar{c}_j = c_j - \pi A_{.j}$ = relative cost coeffs. wrt x_B, are all ≥ 0.
Optimal	If it is both primal and dual feasible.

So, dual feasibility cond. for a basic vector, same as Opt. cond. in primal simplex algo.

Example

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b
1	2	3	-2	1	16	0	17
0	1	-4	1	1	1	0	2
0	0	1	-2	1	0	0	1
3	11	-15	10	4	57	1	$= 0$

$x_j \geq 0$ for all j, min z.

$$(x_1, x_2, x_3), \quad \text{Inverse tableau} = \begin{pmatrix} 1 & -2 & -11 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -5 & 4 & 1 \end{pmatrix}$$

$$(x_4, x_5, x_6), \quad \text{Inverse tableau} = \begin{pmatrix} 1/48 & 1/3 & -5/16 & 0 \\ 1/16 & 0 & -1/16 & 0 \\ -1/24 & 2/3 & 3/8 & 0 \\ -51/16 & -6 & -83/16 & 1 \end{pmatrix}$$

To Check Optimality of a Given Feasible Solution

Say $\bar{x}$. Write all equality constraints that dual variables have to satisfy, if $\bar{x}$ were to be primal opt., from the C.S. conds. If these equality constraints have a unique sol., $\bar{\pi}$ say, and it satisfies all other dual constraints, then $\bar{x}, \bar{\pi}$ are respectively primal and dual opt.

Example: Check whether $\bar{x} = (12, 7, 2, 1, 0)^T$ is opt. to following LP.

x_1	x_2	x_3	x_4	x_5	b
1	-2	2	1	3	3
0	1	-1	-2	4	3
0	0	2	5	-1	9
-6	9	-5	10	-25	$= z(x)$, minimize

x_1 to $x_5 \geq 0$

Example: Check whether $\bar{x} = (10, 5, 0, 0, 0)^T$ is opt. to following LP.

x_1	x_2	x_3	x_4	x_5	b
2	1	-1	2	1	25
2	0	2	-1	3	20
8	6	-10	20	-2	$= z(x)$, minimize

x_1 to $x_5 \geq 0$

Example: Check whether $\bar{x} = (10, 15, 0, 0, 0)^T$ is opt. to following LP.

x_1	x_2	x_3	x_4	x_5	b
1	0	1	-2	1	10
0	1	-1	2	2	15
1	1	1	1	-1	25
1	4	-2	4	20	$= z(x)$, minimize

x_1 to $x_5 \geq 0$

Theorem: If LP in standard form has a nondegenerate opt. BFS, dual optimum is unique, and it is the *marginal value vector* for the primal.

Theorem: Any algorithm for solving linear inequalities (without any optimization) can be used to solve an LP directly.

How to Check if an Opt. Sol. to LP Is Unique?

Consider LP in standard form: min $z = cx$, subject to $Ax = b, x \geq 0$, where $A_{m \times n}$ and rank m.

Suppose $\bar{x}$ is an opt. sol. If $\bar{x}$ is not a BFS, obviously, the problem has alternate optima. All feasible solutions obtained during purification process to obtain a BFS from $\bar{x}$ are alternate optima.

Suppose $\bar{x}$ is a BFS. Let $x_B = (x_1, \ldots, x_m)$ be an optimum basic vector corresponding to $\bar{x}$.

If relative cost coeffs. of nonbasics $\bar{c}_{m+1}, \ldots, \bar{c}_n$ are all positive, by C.S. theorem, $\bar{x}$ is the unique primal optimum solution.

If some nonbasic rel. costs, say $\bar{c}_{m+1}, \ldots, \bar{c}_{m+r}$ are zero, and the others are positive. By C.S. Theorem, any feasible solution with $x_{m+r+1} = \ldots = x_n = 0$ is optimum to the primal. So bringing any of the nonbasics $x_{m+1}, \ldots, x_{m+r}$ into x_B leads to an alternate opt. BFS if the pivot step is nondegenerate.

To check if an alternate optimum exists, we could fix $x_{m+r+1}, \ldots, x_n$

all at 0 and delete them. In the remaining system, maximize $x_{m+1} + \ldots + x_{m+r}$ beginning with x_B. If maximum is 0, $\bar{x}$ is the unique primal opt. Otherwise, the optimum solution of this problem is an alternate opt. for primal.