3.1

Polyhedral Geometry

Let $K \subset \mathbb{R}^n$ be a convex polyhedron.

Extreme or Corner Point, or Vertex: GEOMETRIC DEFINITION: $\bar{x} \in K$ is said to be an extreme or corner point or vertex of $K$ if for any $0 < \alpha < 1$

$$x^1, x^2 \in K, \quad \bar{x} = \alpha x^1 + (1 - \alpha)x^2 \implies x^1 = x^2 = \bar{x}$$

If $K$ is specified by a system of linear constraints, checking efficiently whether a point in it is an extreme point, boils down to checking the linear independence of a set of vectors.

Active, Inactive Constraints at a feasible solution $\bar{x}$: Let $K$ be the set of feasible solutions of the following system of constraints:

$$A_i x = b_i \quad \text{for } i = 1 \text{ to } p$$

$$\geq b_i \quad \text{for } i = p + 1 \text{ to } m$$

**Equality constraints are always active** at every feasible solution $\bar{x} \in K$. 
For $i = p + 1$ to $m$, $i$th constraint (inequality)

**active** at $\bar{x} \in K$ if it holds as an eq. at $\bar{x}$, i.e., if $A_i \bar{x} = b_i$

**inactive** or **slack** at $\bar{x} \in K$ if it holds as a strict ineq. at $\bar{x}$, i.e., if $A_i \bar{x} > b_i$.

Basic Feasible Solution (BFS) for a System of Linear Constraints: ALGEBRAIC DEFINITION

Let $(P)$ be a system of linear constraints, including possibly equations, inequalities, and bounds on variables.

A **feasible solution** for $(P)$ is a vector that satisfies all constraints in $(P)$.

**BFS for $(P)$**: Let $\bar{x}$ be a feasible solution for $(P)$. Let $(S)$ be the system of linear equations obtained by treating all active constraints at $\bar{x}$ as equations. $\bar{x}$ is said to be a BFS for $(P)$, iff it is the unique solution for the active system $(S)$ treated as linear equations.

**Nondegenerate, Degenerate BFSs**: When $\bar{x}$ is a BFS for
(P), it is said to be a **nondegenerate BFS** if the system of equality constraints (S) defined above is a square system (i.e., the number of equations in it = the number of variables in it); **degenerate BFS** if the number of equations in (S) is more than the number of variables in it.

**Theorem:** Let \( K \) be the set of feasible solutions of the system of linear constraints (P). A feasible solution \( \tilde{x} \in K \) is an extreme point of \( K \) iff it is a BFS for (P).

**Example:**

\[
\begin{array}{ccccc}
 x_1 & x_2 & x_3 & x_4 & x_5 \\
 \hline
 1 & 1 & 4 & 12 & 2 & 16 \\
 0 & 1 & 1 & 3 & -4 & 4 \\
 \end{array}
\]

\( x_j \geq 0 \) for all \( j \)

\( x^1 = (12, 4, 0, 0, 0)^T, \quad x^2 = (0, 0, 1, 1, 0)^T. \)

**Nondegenerate, Degenerate systems, polyhedra:** Let \( K \) be the set of feasible solutions of a system of linear constraints (P). \( K, (P) \) are said to be **nondegenerate** if every BFS is nondegenerate; **degenerate** if at least one BFS is degenerate.
BFS for Standard Form: A feasible solution $\bar{x}$ for system

$$Ax = b$$

$$x \geq 0$$

is a BFS iff $\{A_{j} : j \text{ such that } \bar{x}_{j} > 0\}$ is linearly independent.

Nondegenerate, Degenerate BFSs for Standard Form:

For system in standard form $Ax = b, x \geq 0$ where $A_{m \times n}$ has rank $m$; a BFS $\bar{x}$ is nondegenerate if the number of positive variables in it is $m$, degenerate if this number is $< m$.

So, for a nondegenerate BFS the positive variables in it define the unique basic vector, basis matrix corresponding to it.

A basic vector corresponding to a degenerate BFS always contains some 0-valued basic variables, these can be chosen among the 0-valued variables in the BFS arbitrarily as long as the linear independence condition holds. So, usually, a degenerate BFS corresponds to many basic vectors.

Example:
\[ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \]
\[
\begin{array}{ccccc}
1 & 0 & 3 & 1 & 2 & 6 \\
0 & 1 & 4 & 2 & 1 & 8 \\
\end{array}
\]

\[ x_j \geq 0 \text{ for all } j \]

\[ \bar{x} = (6, 8, 0, 0, 0)^T, \quad \tilde{x} = (0, 0, 2, 0, 0)^T. \]

**BFS of Standard Form for Bounded Variable System:**

A feasible solution \( \bar{x} \) for system

\[
A \bar{x} = b
\]

\[
\ell = (\ell_j) \leq x \leq u = (u_j)
\]

is a BFS iff \( \{ A_j : j \text{ such that } \ell_j < \bar{x}_j < u_j \} \) is linearly independent.

In the above system if \( A_{m \times n} \) has rank \( m \), a BFS \( \bar{x} \) for it is nondegenerate if \( J = \{ j : \ell_j < \bar{x}_j < u_j \} \) has cardinality \( m \); degenerate if \( |J| < m \).
Purification Routine to Derive a BFS from a Feasible Solution

We describe for the system in standard form, \( Ax = b, \ x \geq 0 \) where \( A_{m \times n} \). Let \( \bar{x} \) be a feasible solution with \( J = \{ j : \bar{x}_j > 0 \} \).

If \( \Gamma = \{ A_{j} : j \in J \} \) is linearly independent, \( \bar{x} \) is a BFS, terminate.

If \( \Gamma \) linearly dependent, let the \( \ell.d. \) relation be

\[
\sum_{j \in J} \alpha_j A_{j} = 0
\]

we also have \( \sum_{j \in J} \bar{x}_j A_{j} = b \)

So \( \sum_{j \in J} (\bar{x}_j + \lambda \alpha_j) A_{j} = b \)

where \( \lambda \) is a real valued parameter. Hence if we define \( x(\lambda) = (x_j(\lambda)) \) where

\[
x_j(\lambda) = \begin{cases} \bar{x}_j + \lambda \alpha_j & j \in J \\ 0 & j \notin J \end{cases}
\]

then \( x(\lambda) \) satisfies \( Ax = b \). Let
\[ \theta_1 = \begin{cases} \infty & \text{if } \alpha_j \leq 0 \text{ for all } j \in J \\ \max\{-\bar{x}_j/\alpha_j : j \in J, \alpha_j > 0\} & \text{otherwise} \end{cases} \]
\[ \theta_2 = \begin{cases} \infty & \text{if } \alpha_j \geq 0 \text{ for all } j \in J \\ \min\{-\bar{x}_j/\alpha_j : j \in J, \alpha_j < 0\} & \text{otherwise} \end{cases} \]

Then \( \theta_2 > 0 > \theta_1 \) and at least one of them is finite. For all \( \theta_1 \leq \lambda \leq \theta_2 \), \( x(\lambda) \) is a feasible solution.

Let \( \theta = \theta_1 \) or \( \theta_2 \) whichever is finite. Then \( x(\theta) \) is a feasible solution in which number of positive variables is one less than that of \( \bar{x} \). Repeat with \( x(\theta) \).

**Example**

\[
\begin{array}{ccccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
1 & 0 & 1 & -3 & -6 & 12 & 2 \\
-1 & 2 & 1 & -1 & 17 & 18 & 10 \\
0 & 1 & 1 & -2 & 8 & -5 & 6 \\
0 & 1 & 1 & -2 & -9 & -6 & 6 \\
\end{array}
\]

\[ x_j \geq 0 \text{ for all } j \]

\( \bar{x} = (1, 2, 10, 3, 0, 0)^T. \)

**Example:** Consider \( \bar{x} = (1/3, 4/3)^T \) feasible to
\[ x_1 + 2x_2 \geq 2 \]
\[ 2x_1 + x_2 \geq 2 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
\[ -x_1 + x_2 \leq 2 \]

**Theorem:** If system in standard form \( Ax = b, x \geq 0 \) has a feasible solution, it has a BFS.

**Theorem:** If LP in standard form

\[
\begin{align*}
\min \quad z &= cx \\
\text{subject to} \quad Ax &= b \\
x &\geq 0
\end{align*}
\]

has an optimum solution, it has one which is a BFS.
Adjacency of Extreme Points, Edges

Adjacency: GEOMETRIC DEFINITION: Two extreme points $x^1, x^2$ of a convex polyhedron $K \subset \mathbb{R}^n$ are said to be adjacent iff every point $\bar{x}$ on the line segment joining them satisfies:

$$x^3, x^4 \in K, \bar{x} = \alpha x^3 + (1 - \alpha) x^4 \text{ for some } 0 < \alpha < 1 \implies x^3, x^4 \text{ are also on the line segment joining } x^1, x^2.$$

ALGEBRAIC DEFINITION: Suppose $K$ is specified by system of linear constraints $(P)$. Let $(S)$ be the system of linear equations obtained as follows: $(S)$ contains all the linear equations in $(P)$, and all the inequality constraints and bound restrictions which hold as equations at some point in the interior of the line segment joining $x^1, x^2$, for example $(x^1 + x^2)/2$, treated as equations. $x^1, x^2$ are adjacent iff the set of solutions of $(S)$ is the straight line joining $x^1, x^2$.

Theorem: Two BFSs $x^1, x^2$ of system in standard form $Ax = b, x \geq 0$ are adjacent iff rank of $\{A_j: j\}$ such that $\bar{x}_j > 0$, where
\( \bar{x} \) is some point in the interior of the line segment joining \( x^1, x^2 \), for example \( (x^1 + x^2)/2 \) is one less than its cardinality.

**Example:** Put following system in standard form, & check whether \( x^1, x^2 \) are adjacent extreme points of it. What about \( x^1, x^3 \)?

\[
\begin{align*}
  x_1 + 2x_2 & \geq 2 \\
  2x_1 + x_2 & \geq 2 \\
  x_1 & \geq 0 \\
  x_2 & \geq 0 \\
  -x_1 + x_2 & \leq 2
\end{align*}
\]

\( x^1 = (0, 2)^T, x^2 = (2/3, 2/3)^T, x^3 = (2, 0)^T. \)

**Bounded Edge:** An *(Bounded) edge* of a convex polyhedron is the line segment joining two adjacent extreme points.

**How to check whether \( \bar{x} \) feasible to a system \((P)\) is on an edge, and whether that edge is bounded or unbounded**
If \( \bar{x} \) is a BFS (Extreme point) it is of course on edges containing it. So assume \( \bar{x} \) not a BFS.

Let \((S)\) be the active system of constraints at \( \bar{x} \) treated as a system of eqs. \( \bar{x} \), which is not a BFS, is on an edge iff set of solutions of \((S)\) is one dimensional, i.e., is a straight line.

If \( \bar{x} \) on an edge, that edge bounded iff set of sols. of \((S)\) \& all other constraints in \((P)\) not in \((S)\) is a line segment; unbounded edge otherwise.

Example: Consider \( \bar{x} = (5, 10, 3, 0, 0)^T \) feasible to

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>-9</td>
</tr>
<tr>
<td>( x_j \geq 0 ) for all ( j )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example: Consider \( \bar{x} = (2, 4, 8, 12, 0, 0)^T \) feasible to
\[
\begin{array}{cccccc}
\hline
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\hline
1 & 0 & 0 & -1 & 1 & 17 \\
0 & 1 & -1 & 0 & 2 & 18 \\
0 & 0 & 1 & -1 & 3 & 19 \\
\hline
\end{array}
\]

\[x_j \geq 0 \text{ for all } j\]

Example: Consider \(\bar{x} = (2, 4, 8, 12, 0, 0)^T\) feasible to

\[
\begin{array}{cccccc}
\hline
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\hline
1 & 0 & 0 & 1 & 1 & 17 \\
0 & 1 & -1 & 2 & 2 & 18 \\
0 & 0 & 1 & -1 & 3 & 19 \\
\hline
\end{array}
\]

\[x_j \geq 0 \text{ for all } j\]
How to Obtain Adjacent BFSs?, Pivot Steps

Let $K$ be the set of feasible solutions of system in standard form, $Ax = b, x \geq 0$ where $A_{m \times n}$ and rank $m$.

A basic vector for this system is said to be a feasible basic vector, if corresponding basic solution is $\geq 0$, i.e., it is a BFS.

Let $x^1$ be a BFS associated with basic vector $x_B$, basis $B$. Let $x_D$ be vector of nonbasic variables. System can be rearranged into basic, nonbasic parts.

<table>
<thead>
<tr>
<th>Original Tableau</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_B$</td>
</tr>
<tr>
<td>$B$</td>
</tr>
</tbody>
</table>

$x \geq 0$

The canonical tableau for system wrt $x_B$ obtained by multiplying tableau on left by $B^{-1}$.

<table>
<thead>
<tr>
<th>Canonical Tableau</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic var.</td>
</tr>
<tr>
<td>$x_B$</td>
</tr>
</tbody>
</table>

$x \geq 0$
$x^1$, the BFS wrt $x_B$ is given by:

Nonbasic variables $x_D = 0$

Basic vector $x_B = B^{-1}b = \bar{b} \geq 0$

Process of obtaining an adjacent BFS of $x^1$ starts with the following:

(i) Select one nonbasic variable, $x_s$ say, as the entering variable into the basic vector $x_B$.

(ii) Fix all nonbasic variables other than $x_s$ at 0.

(iii) Make $x_s$, the entering nonbasic = $\lambda$, a parameter.

(iv) Find unique values of basic variables in $x_B$ as functions of $\lambda$

to satisfy $Ax = b$.

We first show these before describing the rest of the process. The whole process is called pivot step for entering $x_s$ into $x_B$. Below we show remaining canonical tableau after doing (ii).
So \( x(\lambda) \) is given by the following:

\[
\begin{align*}
\text{Other nonbasics} & = 0 \\
\text{Entering} \ x_s & = \lambda \\
\text{Basic} & = \\
\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} & = \\
\begin{pmatrix} \bar{b}_1 - \lambda \bar{a}_{1s} \\ \vdots \\ \bar{b}_m - \lambda \bar{a}_{ms} \end{pmatrix}
\end{align*}
\]

This solution remains feasible if \( \bar{b}_i - \lambda \bar{a}_{is} \geq 0 \) for all \( i \), i.e., if \( 0 \leq \lambda \leq \theta \) where

\[
\theta = \begin{cases} 
+\infty & \text{if } \bar{a}_{is} \leq 0 \text{ for all } i \\
\min\{\bar{b}_i/\bar{a}_{is} : \text{over } \bar{a}_{is} > 0\} & \text{otherwise}
\end{cases}
\]

This \( \theta \) is called \textit{minimum ratio} in this pivot step, process of computing it called \textit{minimum ratio test}.

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1. If \( \theta = \infty \), \( \{x(\lambda) : \lambda \geq 0\} \) is an unbounded edge or extreme half-line of \( K \). Its direction given by:

\[
\begin{align*}
\text{Other nonbasics} & = 0 \\
\text{Entering } x_s & = 1 \\
\text{Basic} & = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} -\bar{a}_{1s} \\ \vdots \\ -\bar{a}_{ms} \end{pmatrix}
\end{align*}
\]

is an extreme direction of \( K \). It is an extreme point of the normalized homogeneous system:

\[
Ax = 0 \\
\sum_{j=1}^{n} x_j = \beta \\
x \geq 0
\]

for some strictly positive quantity \( \beta \).

2. If \( 0 < \theta < +\infty \), \( x(\theta) \) is an adjacent BFS of \( x^1 \). In \( x(\theta) \) at least one present basic variable (one attaining min in the min ratio
test) is 0, select one of them as the **dropping basic variable** to be replaced by entering variable, leading to a new feasible basic vector, $x_{\tilde{B}}$ say, for which $x(\theta)$ is the BFS.

If dropping basic variable is $r$th, performing pivot step with updated col of $x_s$ as pivot col and $r$th row as pivot row in present canonical tableau, leads to canonical tableau wrt $x_{\tilde{B}}$.

Purpose of min ratio test: To determine dropping basic var. that entering var. should replace to make sure next basic vector is also feasible.

When min ratio $\theta > 0$ and finite, pivot step called **nondegenerate pivot step**. It always leads to an adjacent BFS. The line segment joining $x^1$ and $x(\theta)$ is an **edge (bounded edge)** of $K$.

**3.** If $\theta = 0$, $x(\theta) = x^1$. So, new basic vector $x_{\tilde{B}}$ is another basic vector corresponding to BFS $x^1$. In this case pivot step called **degenerate pivot step**. In a degenerate pivot step, BFS does not change, but basic vector changes.

**Example:** Consider basic vector $(x_1, x_2, x_3)$ for
Example: Consider extreme point $x^1 = (1, 1)^T$ for the system:

\[
\begin{align*}
&x_1 + x_2 \geq 2 \\
&x_1 - x_2 \geq 0 \\
&x_1, x_2 \geq 0
\end{align*}
\]
The Main Results

1. Consider the LP: to minimize $z = cx$ subject to some linear constraints.

   Let $K$ denote set of feasible sols.

   Let $\bar{x}$ be an extreme point of $K$ satisfying:

   Moving away from $\bar{x}$ along any edge incident at $\bar{x}$ either increases the value of $z = cx$, or keeps it unchanged.

   Then $\bar{x}$ is an optimum solution of this LP.

2. Consider LP in standard form: min $z = cx$ subject to $Ax = b$, $x \geq 0$.

   The minimum value of $z$ in this LP is $-\infty$ (i.e., $z$ unbounded below in this LP) iff there exists a BFS $\tilde{x}$, and an unbounded edge incident at $\tilde{x}$ along which $z$ diverges to $-\infty$.

   These results are the foundation for the Simplex Algorithm for LP.
4.17

Faces of a Convex Polyhedron

Let $K \subset \mathbb{R}^n$ be a convex polyhedron.

Supporting Hyperplane for $K$: A hyperplane $H$ is said to be a supporting hyperplane for $K$ if $H \cap K \neq \emptyset$ and $K$ is completely contained on one side of $H$.

Face of $K$: A face of $K$ is either $\emptyset$, or $K$ itself, or the intersection $H \cap K$ for some supporting hyperplane $H$ of $K$.

Faces of a Polyhedron Defined by Linear Constraints: Let $K$ be the polyhedron by system of linear constraints $(P)$.

Take a subset of inequality constraints in $(P)$ and make them into equations. Take a subset of variables with bound restrictions and make each of them $=$ one of the bounds on it. Let $(Q)$ be the resulting system.

The set of feasible solutions of $(Q)$ is a face of $K$, and conversely every face of $K$ is the set of feasible solutions of a system like $(Q)$ obtained from $(P)$. 
Faces for System in Standard Form: Let $K$ be set of feasible solutions of $Ax = b, x \geq 0$. Each face of $K$ is the set of feasible solutions of remaining system after a subset of variables is fixed at 0.

Theorem: Let $K$ be the set of feasible solutions of an LP. The set of optimum solutions of this LP is always a face of $K$.

The extreme points of a convex polyhedron are its 0-dimensional faces. The 1-dimensional faces are the edges. The vertices and edges put together define the one dimensional skeleton or the graph of the polyhedron.

Faces of the polyhedron whose dimension is one less than the dimension of the polyhedron are called its facets.
Boundedness of Convex Polyhedra

Homogeneous System: Let \((P)\) be a system of linear constraints. The system \((H)\) obtained by changing all the RHS constants in all constraints and bound restrictions in \((P)\) to 0, is called the homogeneous system corresponding to \((P)\).

Theorem: Every feasible solution \(\bar{x}\) of the system in standard form: \(Ax = b, x \geq 0\) can be expressed as:

(a convex comb. of BFSs) + (a homogeneous solution).

Extreme Homogeneous Solution: The homogeneous system corresponding to standard form above is: \(Ay = 0, y \geq 0\).

An extreme homogeneous solution is an extreme point of the normalized homogeneous system, i.e.,

\[
Ay = 0 \\
\sum y_j = 1 \\
y \geq 0
\]
Theorem: Every homogeneous solution for system in standard form is a nonnegative comb. of extreme homogeneous solutions.

Theorem: If $K = \text{set of feasible solutions of system in standard form}$, and $K \neq \emptyset$, $K$ bounded iff 0 is the unique homogeneous solution.

Theorem: Resolution Theorem for Systems in Standard Form: Every feasible solution can be expressed as

(a convex comb. of BFSs) + (a nonnegative comb. of extreme homogeneous solutions).

Theorem: The LP in standard form

$$\text{Min } z = cx$$

subject to $Ax = b$

$x \geq 0$

has unbounded minimum (i.e., $z \to -\infty$) iff it is feasible, and
the following system is feasible in $y$

\[ Ay = 0 \]
\[ cy < 0 \]
\[ y \geq 0 \]