2.1

Special Geometric Method to Solve 2 Var. LPs, Planning Uses of LP Output

Linear Hull or *Subspace* of $\{A_1, \ldots, A_k\} \subset R^n$: is $\{x : x = \alpha_1 A_1 + \ldots \alpha_k A_k, \quad \alpha_1, \ldots, \alpha_k$ take all real values $\}$.

Set of solutions of a homogeneous system of form $Dx = 0$ is a subspace.

Affine Hull or *Affine Space* of $\{A_1, \ldots, A_k\} \subset R^n$: is $\{x : x = \alpha_1 A_1 + \ldots \alpha_k A_k, \quad \alpha_1, \ldots, \alpha_k$ take all real values satisfying $\alpha_1 + \ldots + \alpha_k = 1$ $\}$.

Set of solutions of a system of linear equations of form $Dx = d$ is an affine space.

Affine space of two points is the straight line joining them.

Convex Hull of $\{A_1, \ldots, A_k\} \subset R^n$: is $\{x : x = \alpha_1 A_1 + \ldots \alpha_k A_k, \quad \alpha_1, \ldots, \alpha_k$ take all NONNEGATIVE values satisfying $\alpha_1 + \ldots + \alpha_k = 1$ $\}$.

Set of solutions of a system of linear equations and inequalities of form $Dx = d, \quad Ex \geq f$ is closed under the operation of taking convex combinations

Convex hull of two points is the line segment joining them.

Nonnegative Hull or *Pos Cone* of $\{A_1, \ldots, A_k\} \subset R^n$: is $\{x : x = \alpha_1 A_1 + \ldots \alpha_k A_k, \quad \alpha_1, \ldots, \alpha_k$ take all nonnegative values $\}$. Also denoted by $Pos\{A_1, \ldots, A_k\}$.

Set of solutions of a system of homogeneous linear inequalities of form $Dx \geq 0$ is a Pos cone.

If A is a matrix, $Pos(A)$ denotes the nonnegative hull of its set of column vectors, a convex polyhedral cone.

Ray of a point $a \in R^n$: is $\{\alpha a : \alpha \geq 0\}$. It is the halfline joining 0 to a and continuing indefinitely in that direction.

Half-Line through b *parallel to the ray of* a: is $\{b + \alpha a : \quad \alpha \geq 0\}$. b is its initial point, and a is said to be its direction.

Moving from b in the direction of a: means traveling from b along the half-line through b parallel to the ray of a.

Hyperplane in R^n: Set of solutions of a single nontrivial equation, i.e., $\{x : a_1x_1 + \ldots + a_nx_n = b_1$ where at least one of $a_1, \ldots, a_n \neq 0\}$.

In R^2 every hyperplane is a straight line and vice versa.

Half-Space in R^n: Set of solutions of a single nontrivial inequality, i.e., $\{x : a_1x_1 + \ldots + a_nx_n \geq b_1$ where at least one of

$a_1, \ldots, a_n \neq 0\}$.

Each hyperplane divides the space into two half-spaces whose intersection is the hyperplane.

Convex Set: A set K satisfying: $x^1, x^2 \in K \Rightarrow \alpha x^1 + (1 - \alpha)x^2 \in K \quad$ for all $0 \leq \alpha \leq 1$.

Convex Polyhedral Set: Intersection of a finite number of half-spaces.

Convex Polytope: A bounded convex polyhedron.

Convex Polyhedral Cone: Intersection of a finite number of half-spaces, each containing the origin.

Geometric Method to Solve 2 Variable LPs

1. Set up the axes of coordinates. Identify the half-space corresponding to each constraint, and bound restriction. *Feasible region* K is the intersection of all these half-spaces.

If $K = \emptyset$, problem *infeasible*, terminate.

2. If $K \neq \emptyset$, select a point $\bar{x} \in K$ and draw the objective line $z(x) = \bar{z} = z(\bar{x})$.

Move objective line parallel to itself by increasing (*decreasing*) the RHS constant in its equation from $\bar{z}$ if $z(x)$ is to be maximized (*minimized*), always keeping its intersection with K nonempty. Two possibilities.

(a) The line can be moved indefinitely, keeping its intersection with K nonempty always. In this case $z(x)$ is unbounded above (*below*) on K.

(b) z_0 is the last value of RHS constant in above process. Any further change in it, makes the objective line loose its intersection with K.

In this case z_0 is the optimum value of $z(x)$ and all the points in $K \cap \{x : z(x) = z_0\}$ are optimum solutions of the problem.

Example: Fertilizer Problem

Points to Remember: This method not extended to higher dimensions. There objective line becomes objective hyperplane, and we cannot get visual information whether it can be moved while still keeping nonempty intersection with feasible region.

Path followed by simplex method very different in character.

THEOREM: If an LP in standard form has an optimum solution, it has one which is a basic solution of the system of equality constraints.

Planning Information From LP Models

Optimum Solution: Solving model gives it, if one exists.

Infeasibility Analysis: If model infeasible, this helps in making necessary changes to make it feasible.

Identify Critical Items: Items corresponding to slacks $= 0$ in opt. sol. are at their bound, hence critical. Other items are not.

Marginal Values: The marginal value of an item (or the RHS constant in the corresponding constraint) is the rate of change in opt. obj. value per unit change in that RHS constant.

When model solved, we can determine

- Whether marginal values exist or not.
- Their values, if they exist.

Example: Fertilizer Problem.

Theorem: For LP in standard form, if opt. sol. is the *non-degenerate basic solution* of system of equality constraints associated with basis B, then marginal value vector exists, and it is $\pi = c_B B^{-1}$ where c_B is the row vector of objective coefficients of variables associated with columns in B.

Marginal values have many planning applications.

Example: Evaluating Profitability of New Products

New fertilizer *Lushlawn*. Per ton made, needs inputs:

$$3 \text{ tons } RM1$$

$$2 \text{ tons } RM2$$

$$2 \text{ tons } RM3$$

How much profit should lushlawn fetch per ton made, before company considers manufacturing it?