1.1

Formulation Techniques

Katta G. Murty Lecture slides

How To Build a Mathematical Model?

1. Identify All Decision Variables

Controllable parameters whose values can be controlled by decision maker, which affect functioning of system. Denote them by $x_1, \ldots, x_n$.

$$x = \quad \text{Decision vector} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \quad (x_1, \ldots, x_n)^T$$

2. Identify Objective Function and All Constraints on Decision Variables

$$g_i(x) = b_i \quad \text{An Equality Constraint}$$

$$\left.\begin{array}{l} g_i(x) \geq b_i \\ g_i(x) \leq b_i \end{array}\right\} \quad \text{Inequality Constraints}$$

Constraint Functions, Right Hand Side (RHS) Constants.

$x_j \geq b_j$ or $x_j \leq b_j$	Lower or Upper bound constraints on individual variables.
$x_j \geq 0$	Lower bound constraint called *Non-negativity restriction.*

Objective function called $\begin{cases} \textit{COST FUNCTION} & \text{if to be min.} \\ \textit{PROFIT FUNCTION} & \text{if to be max.} \end{cases}$

Some Definitions

LINEAR FUNCTION: One of form $c_1x_1 + \ldots + c_nx_n$

where $c = (c_1, \ldots, c_n)$ is *coefficient vector* of variables in it.

Example: $x \in R^4$. $x_2 - 7x_4$ is a linear function with coefficient vector $(0, 1, 0, -7)$.

AFFINE FUNCTION: A linear function + a constant, i.e., One of form $c_0 + cx$.

FEASIBLE SOLUTION: A vector x that satisfies all the constraints.

OPTIMUM SOLUTION: A feasible solution that gives the best value for

objective function among all feasible solutions.

LINEAR PROGRAM: Optimization problem in which objective
function and all constraint functions are linear.

Steps in Modeling A Linear Program

1. LIST ALL DECISION VARIABLES: Each decision variable is the level at

   which an *ACTIVITY* is carried out.

2. VERIFY LINEARITY ASSUMPTIONS: *Proportionality Assumption*

   and *Additivity Assumption.* Must hold for

   objective function and all constraint functions.

3. VERIFY ALL VARIABLES ARE CONTINUOUS VARIABLES:

4. CONSTRUCT OBJECTIVE FUNCTION:

5. IDENTIFY ALL CONSTRAINTS & BOUNDS ON INDIVIDUAL VARIABLES: Each

   constraint is *Material Balance Equation* or *Inequality* of an *ITEM*.

Product Mix Models

Company can make many different products. This determines how many of each to make in a period to maximize profit.

INPUT DATA

1. List of all products that can be made.
2. List of all raw materials and other resources needed as inputs.
3. All *input-output* (or *technology*) coefficients.
4. Cost or net profit coefficients.
5. Bounds on each resource availability.
6. Bounds on amount of each product made.

Example: Fertilizer Company

Two products. 3 raw materials. No limit on amount of each product made.

Table 2.1

Item	Input (tons)/ ton of		Max. available daily (tons)
	Hi-ph	Lo-ph	
RM 1	2	1	1500
RM 2	1	1	1200
RM 3	1	0	500
Net profit ($) / ton	15	10	

Two activities, two decision variables.

Some More Definitions

Slack Variables: RM 1 Constraint $2x_1 + x_2 \leq 1500$ equivalent to $1500 - 2x_1 - x_2 \geq 0$.

Let $s_1 = 1500 - 2x_1 - x_2$, it is *Tons of RM 1 Unused in Solution* x, and called *Slack Variable for RM 1 Constraint.* By introducing it, RM 1 constraint can be written as:

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 1500 \\ s_1 &\geq 0 \end{aligned}$$

1. Every inequality constraint can be written as an equation by introducing appropriate slack variable.

2. The slack variable for each inequality constraint is separate.

3. Sign of slack in equation is +1 or -1 depending on whether inequality is $\leq$ or $\geq$.

4. Slacks are always nonnegative variables.

5. LP said to be in *STANDARD FORM* if all constraints are equations and all variables are nonnegative variables. Every LP can be transformed into standard form.

Production with Environmental Protection

Firm produces chemical **C** using two main raw materials **A, B**.

1 lb.**A** + 2 lb.**B** $\rightarrow$ $\boxed{\text{Process 1}}$ $\rightarrow$ 1 lb each of **C, LW, SW**

LW = Liquid Waste, currently dumped in river.

SW = Solid Waste, currently being taken by a fertilizer company at no charge, no payment.

From next year, direct dumping of **LW** in river will be prohibited by EPA. 3 alternatives for handling **LW**.

Alternative 1: Treat **LW** (cost \$0.35/lb) to remove pollutants, then release into river.

Alternative 2: 1 lb **A** + 1 lb **LW** $\longrightarrow$ $\boxed{\text{Process 2}}$ $\longrightarrow$ 2 lb of byproduct **D**

Alternative 3: 1 lb **B** + 1 lb **LW** $\longrightarrow$ $\boxed{\text{Process 3}}$ $\longrightarrow$ 2 lb of byproduct **E**

Costs	
1 lb **A**	\$1.5, upto 5000 lb **A** available/day
1 lb **B**	\$1.75, upto 7000 lb **B** available/day
Process 1 labor/lb **C** made	\$0.5
Process 1 other costs/lb **C** made	\$1.6
Process 2 labor/lb of **D** made	\$0.2
Process 3 labor/lb of **E** made	\$0.15
Treatment/lb of **LW**	\$0.45

Product	Selling Price
C	\$5.95/lb
D	\$0.85/lb
E	\$0.65/lb

Required to find best plan for company while meeting EPA guidelines.

Linear Blending Models

Many applications in Petroleum, Food, Paint, Chemical, Pharmaceutical industries.

Linearity Assumptions: Suppose $p_1 + \ldots + p_k = 1$ and a mixture contains:

Materials	M_1	M_2	$\ldots$	M_k
In Proprtion	p_1	p_2	$\ldots$	p_k
Characteristic t for material	t_1	t_2	$\ldots$	t_k

Linearity holds iff Characteristic for mixture is: $\Sigma_{r=1}^{k} p_r t_r$.

Example: Mixture, 4 barrels of fuel F_1 and 6 barrels of fuel F_2. OCR of F_1, F_2 are 68, 92 respectively. What will be the OCR of mixture under linearity?

Decision Variables in Blending Models Are

Either QUANTITIES of materials blended, $w_1, \ldots, w_k$, say. In this case remember to include constraint that: Quantity of blend made, $w = w_1 + \ldots + w_k$.

Or PROPORTIONS of materials in the blend, $p_1, \ldots, p_k$, say. In this case remember to include the constraint that: $p_1 + \ldots + p_k = 1$.

Gasoline Blending Example: Specs. on many properties: Octane Rating (OCR), Viscosity, Vapor Pressure, Pour Point, Freezing Point, etc. For simplicity we consider only one property, OCR.

Company blends 4 raw gasolines RG_1 to RG_4 into 3 different grades of fuel F_1 to F_3. All quantities measured in B (Barrels).

Any RG not blended into fuels can be sold to small dealers at:

$38.95/$B$. if OCR > 90

$36.85/$B$. if OCR ≤ 90

Other data given below. Formulate best buying, blending, selling plan.

RG	OCR	Max. Availability/day	Cost \$/B.
RG_1	68	4000 B.	$31.02
RG_2	86	5050	33.15
RG_3	91	7100	36.35
RG_4	99	4300	38.75
Fuel	**Min. OCR**	**Price \$/B.**	**Demand**
F_1	95	$45.15	$\leq$10,000 B. /day
F_2	90	42.95	No limit
F_3	85	40.99	$\geq$ 15,000 B./day

The Diet Model

Nutriets	Chemicals that body needs daily to stay healthy and fit. EXAMPLES: Vitamins A, B, C, D, E, etc., different proteins, starches, etc., fiber; etc.
Foods	Available in retail stores. EXAMPLES: Milk (measured in gallons), yogurt (cups), rice (pounds), bread (slices), lettuce (heads), bean curd (packets), etc.
Diet	Combination of foods that one eats daily. Different foods measured in different units.
Food Composition	No. of units of each nutrient per unit of that food. Determined by a careful chemical analysis.
MDR of nutrient	Minimum Daily Requirement for nutrient. Determined by careful medical analysis.

AIM: Foods vary greatly in cost and composition. Aim is to find a minimum cost diet meeting MDR of all nutrients.

History: First studied by G. J. Stiegler in 1940's. His model had 77 foods at 1939 prices. Combined nutrients into groups (like vitamins, etc.), and considered 9 different nutrient groups. Taste not considered. Modeled as an LP involving 9 constraints in 77 variables. Awarded 1982 ECON. NOBEL PRIZE partly for this model.

Now-A-Days used extensively for designing feed for farm chickens, turkeys, cows, hogs, etc.

EXAMPLE: Two foods, F_1, F_2. Three nutrients: S(starches), P(proteins), V(vitamins).

Nutrient	Nutrient units/kg.		
	F_1	F_2	MDR
S	5	7	8
P	4	2	15
V	2	1	3
Cost ($/kg.)	0.60	0.35	

The Transportation Model

AIM is to find a way of transferring goods at min. transportation cost. In USA alone, transportation of goods is estimated to cost over US $900 billion/year!

SOURCES: Places where material is available.

SINKS, MARKETS, or DEMAND CENTERS: Places where material is required.

AVAILABILITY (at sources): How much material is available at each source.

REQUIREMENTS (at sinks): How much material is required at each sink.

COST COEFFs.: Cost of transporting material/unit from each source to each sink.

BALANCE CONDITION: $\{$Sum availabilities at sources$\}$ = $\{$Sum requirements at sinks$\}$. If this holds, problem called BALANCED TRANSPORTATION PROBLEM.

History: Formulated in 1930's by Russian economist L. V. Kantorovitch. He & T. Koopmans received 1975 ECON. NOBEL PRIZE partly for this work.

EXAMPLE: Sources are 2 mines, sinks are 3 steel plants.

	c_{ij} (cents/ton)			Availability at
	$j = 1$	2	3	mine (tons) daily
Mine $i = 1$	11	8	2	800
2	7	5	4	300
Requirement at plant (tons) daily	400	500	200	

Array Repersentation of Transportation Model

Each row (Column) of array corresponds to a source (Sink). Cell (i, j) in array corresponds to cooresponds to transportation route from source i to sink j. Put variable x_{ij} in its center.

Each row and column of array leads to a constraint on sum of variables in it.

SPECIAL PROPERTY OF TRANSPORTATION MODEL: THE INTEGER PROPERTY: If all availabilities and requirements are positive integers, and if the problem has a feasible solution, then it has an optimum solution in which all variables take only integer values.

Array Representation

	Plant			
	1	2	3	
Mine 1	x_{11} 11	x_{12} 8	x_{13} 2	$= 800$
Mine 2	x_{21} 7	x_{22} 5	x_{23} 4	$= 300$
	$= 400$	$= 500$	$= 200$	

$x_{ij} \geq 0$ for all i, j. Min. cost.

Assignment Problem

It is a transportation problem with equality constraints in which

:

no. of sources = no. of sinks = n say,

all availabilities and all requirements are = 1.

EXAMPLE: Write assignment problem with cost matrix

$$\begin{pmatrix} 5 & 9 & 10 \\ 4 & 5 & 19 \\ 6 & 8 & 8 \end{pmatrix}$$

Has many applications such as in assigning jobs to machines etc.

Marriage application of Assignment Problem.

Halmos and Vaughan (1950) used Assignment problem as a tool to analyze problem of marriage and divorce in society.

Consider club of 5 boys and 5 girls.

c_{ij} = (Mutual) happiness rating of boy i and girl j when they spend a unit time together.

Given $c = (c_{ij})$ determine what fraction of life each boy should spend with each girl, to max. club's happiness.

	c_{ij} for girl $j =$				
	1	2	3	4	5
boy $i = 1$	78	-16	19	25	83
2	99	98	87	16	92
3	86	19	39	88	17
4	-20	99	88	79	65
5	67	98	90	48	60

Multi-Period LP Models

Important applications in production planning for planning production, storage, marketing of product over a planning horizon.

Production capacity, demand, selling price, production cost, may all vary from period to period.

AIM: To determine how much to produce, store, sell in each period; to max. net profit over entire planning horizon.

YOU NEED Variables that represent how much material is in storage at end of each period, and a material balance constraint for each period.

EXAMPLE: Planning Horizon = 6 periods.

Storage warehouse capacity: 3000 tons.

Storage cost: \$2/ton from one period to next.

Initial stock: 500 tons. Desired final stock: 500 tons.

Period	Prod. cost	Prod. capacity	Demand (tons)	Sell price
1	20 \$/ton	1500	1100 tons	180 \$/ton
2	25	2000	1500	180
3	30	2200	1800	250
4	40	3000	1600	270
5	50	2700	2300	300
6	60	2500	2500	320

EXAMPLE:

Company sends 3 air pollutants (particulates (P), sulfur oxides (SO), and hydrocarbons (HC)) from blast furnaces, open hearth furnaces. Each measured in units of million lbs./year. New rules require reductions in these.

Pollutant	Required reductions
P	50
SO	140
HC	130

Emissions can be reduced by 3 techniques. They are:

TS – taller smoke stacks, i.e., increase height.

IF – use filter devices in smokestacks

HG – use more high grade fuels

Following tables give reductions that can be achieved using the method at maximum possible level.

Pollutant	Reduction in blastfurnaces*		
	TS	IF	HG
P	12	25	17
SO	35	20	55
HC	34	27	28
Yearly Cost (in 10^6 \$)+	8	7	11

* using at maximum level

+for maximum level

Pollutant	Reduction in open hearth furnaces*		
	TS	IF	HG
P	9	20	13
SO	40	30	47
HC	50	23	18
Yearly Cost (in 10^6 \$)+	10	6	9

* using at maximum level

+for maximum level

Reductions by 3 techniques are additive. Each can be used at any fraction of its maximum level in any furnace

to get corresponding fractional reduction ,

at corresponding fractional cost.

Determine combination of 3 techniques, as

fractions of the maximum possible levels, to use at each furnace, to

achieve mandated reductions

at minimum cost.

Minimizing Separable Piecewise Linear Objective Functions

Convex and Concave Functions:

A real valued function $f(x) : R^n \to R^1$ is said to be CONVEX iff: For every x^1, x^2 and $0 \leq \alpha \leq 1$

$$f(\alpha x^1 + (1 - \alpha)x^2) \leq \alpha f(x^1) + (1 - \alpha)f(x^2)$$

This inequality called JENSEN'S INEQUALITY.

Separable function:

The function $f(x)$ is said to be separable if it can be written as: $f(x) = f_1(x_1) + \ldots + f_n(x_n)$.

Piecewise Linear Function: A function $\theta(\lambda)$ of a single variable λ, defined on an interval $a \leq \lambda \leq b$ is said to be piecewise linear if it is continuous, and if the interval $[a, b]$ can be divided into a finite number of subintervals in each of which the slope of $\theta(\lambda)$ is constant.

Interval	Slope of $\theta(\lambda)$	$\theta(\lambda)$
a to λ_0	c_1	$\theta(a) + c_1(\lambda - a)$
λ_0 to λ_1	c_1	$\theta(\lambda_0) + c_2(\lambda - \lambda_0)$
	$\vdots$	
λ_k to b	c_{k+2}	$\theta(\lambda_k) + c_{k+2}(\lambda - \lambda_k)$

Example 1

Interval	Slope
0 to 10	3
10 to 25	5
25 to ∞	7

Example 2

Interval	Slope
0 to 100	10
100 to 300	5
300 to 1000	10
1000 to ∞	20

Theorem: A continuous function $\theta(\lambda)$ of a single variable λ is convex (concave) iff its slope is monotonic increasing (monotonic decreasing) with λ.

Important Properties

1. Sum of convex functions is convex.

2. Positive combination of convex functions is convex.

3. Pointwise supremum of convex functions is convex.

Minimizing Separable Piecewise Linear Function Subject To Linear Constraints: Can be transformed into LP iff the objective function is convex.

Technique: Express each variable as a sum of several different new variables; one new variable for each interval of the original variable in which the slope of the objective function is constant.

ORIGINAL VARIABLES: $x_1, \ldots, x_n$.

ORIGINAL OBJECTIVE FUNCTION: $z(x) = z_1(x_1) + \ldots + z_n(x_n)$.

Consider $z_j(x_j)$

Interval of x_j	Slope of $z_j(x_j)$ in interval	Length of interval
0 to x_j^1	c_j^1	$\ell_j^1 = x_j^1$
x_j^1 to x_j^2	c_j^2	$\ell_j^2 = x_j^2 - x_j^1$
	$\vdots$	
x_j^{r-1} to x_j^r	c_j^r	$\ell_j^r = x_j^r - x_j^{r-1}$
x_j^r to ∞	c_j^{r+1}	$\ell_j^{r+1} = \infty$

INTRODUCE NEW VARIABLES: $x_{j1}, x_{j2}, \ldots, x_{j,r+1}$.

RELATIONSHIPS: ORIGINAL $x_j = x_{j1} + x_{j2} + \ldots + x_{j,r+1}$

ORIGINAL OBJ. FUNC. $z_j(x_j) = c_j^1 x_{j1} + c_j^2 x_{j2} + \ldots + c_j^{r+1} x_{j,r+1}$

$$
\text{BOUNDS ON NEW VARIABLES:} \quad
\begin{aligned}
0 &\leq x_{j1} \leq \ell_j^1 \\
0 &\leq x_{j2} \leq \ell_j^2 \\
&\vdots \\
0 &\leq x_{jr} \leq \ell_j^r \\
0 &\leq x_{j,r+1}
\end{aligned}
$$

Example: Company makes products P_1, P_2, P_3. Inputs are: LI (limestone), EP (electric power), W (water), F (fuel), L (labor, measured in man hours). LI, EP, W, F measured in suitable units. Sources for inputs are:

LI Companies own quarry: upto 250 units/day at cost \$20/unit; Outside supplier: any amount at cost \$50/unit.

EP Upto 1000 units/day at \$30/unit, upto 500 more units/day at \$45/unit; amounts beyond 1500 units/day at \$75/unit.

W Upto 800 units at \$6/unit; more at \$7/unit.

F Outside supplier at \$40/unit upto 3000 units/day. No more available.

L Upto 640 mh/day at \$10/mh; upto 160 more mh/day at \$17/mh overtime cost.

Product	Inputs units/unit made				
	LI	EP	W	F	L
P_1	1/2	3	1	1	2
P_2	1	2	1/4	1	1
P_3	3/2	5	2	3	1

Selling prices are:

P_1 \$3000/unit for first 50 units; \$250/unit above 50 units.

P_2 \$3500/unit. No more than 100 units/day.

P_3 \$4500/unit.

Formulate product mix problem to maximize net profit.

Min-Max, Max-Min Models

$$z(x) = \max\{c_0^1 + c^1x, c_0^2 + c^2x, \ldots, c_0^k + c^kx\}$$

To minimize $z(x)$ subject to linear constraints can be transformed into an LP.

TECHNIQUE: Introduce one new variable x_{n+1}. Introduce additional linear constraints:

$$\begin{aligned} x_{n+1} &\geq c_0^1 + c^1 x \\ x_{n+1} &\geq c_0^2 + c^2 x \\ &\vdots \\ x_{n+1} &\geq c_0^k + c^k x \end{aligned}$$

And minimize x_{n+1} subject to all the constraints.

Similarly, maximizing $\min\{c_0^1 + c^1 x, c_0^2 + c^2 x, \ldots, c_0^k + c^k x\}$ can be transformed into an LP.

Example: Application in Worst Case Analysis

Fertilizer problem. Suppose net profits from selling one ton of Hi-Ph, Lo-Ph are unknown. But market analysis established that they will be one of following three:

(15, 10), (10, 15), (12, 12).

Formulate problem of determining product mix that maximizes the worst profit that can be expected.

Minimizing Positive Combinations of Absolute Value Functions

$$z(x) = w_1|c_0^1 + c^1 x| + \ldots + w_k|c_0^k + c^k x|$$

where $w_1, \ldots, w_k$ are all > 0. To minimize $z(x)$ subject to linear constraints can be transformed into an LP.

TECHNIQUE: Express $\quad c_0^t + c^t x = u_t^+ - u_t^-, \quad$ where u_t^+, u_t^- are nonnegative variables.

When $\quad u_t^+ \times u_t^- = 0, \quad |c_0^t + c^t x| = u_t^+ + u_t^-.$

For minimizing $z(x)$, since $w_1, \ldots, w_k > 0, \quad u_t^+ \times u_t^- = 0$ holds automatically.

Example: Meeting Targets As Closely as Possible

Fertilizer problem. Company established following targets:

Daily Net Profit $15x_1 + 10x_2$, target of \$13,000.

Market Share Measured by daily sales revenue, $300x_1 + 175x_2$, target of \$220,000.

Hi-Tech Mkt. Share Measured by daily sales revenue of Hi-Ph, $300x_1$, target of \$80,000.

Weights reflecting priorities of above targets are 10, 6, 8. Formulate problem of finding product mix that minimizes weighted sum of absolute deviations from targets.

Best L_1 or L_∞ Approximations. LP in Curve Fitting

Example: Yield in a chemical reaction depends on temperature t. Following data obtained from experiments.

Temp. t	-5	-3	-1	0	1
Yield, $y(t)$	80	92	96	98	100

Curve Fitting Problem: Develop $y(t)$ as a mathematical function of t, giving best fit to data.

Phase 1: Model Function Selection: Select a mathematical function, $f(t)$ say, with some unknown parameters, $a_0, a_1, \ldots, a_k$, say, that best seems to represent yield.

Phase 2: Parameter Estimation: Find best values of parameters that minimize some measure of deviation between $f(t)$ and $y(t)$ at values of t observed in experiment.

Commonly used measures of deviation

L_2 MEASURE, Sum of squared deviations.

L_1 MEASURE, Sum of absolute deviations.

L_∞ MEASURE, CHEBYSHEV MEASURE, maximum absolute deviation.

Parameter estimation based on L_2 measure called *Least Squares Method*.

If $f(t)$ is linear in the parameters $a_1, \ldots, a_k$, the parameter estimation based on L_1 or L_∞ measures can be transformed into LP.

Residue: Minimum valus of measure of deviation.

If residue reasonably small, accept $f(t)$ with best values for parameters, as the function representing yield.

If residue too large, model function $f(t)$ selected may be a bad choice. Do Phase 1 again.

Example: Fit a polynomial of degree 3 in t for yield data given above by L_1, L_∞ measures.

Minimizing Positive Combinations of Excesses/Shortages

Linear function $\Sigma a_j x_j$ has target value of b, but allowed to take any value.

Excess denoted $(\Sigma a_j x_j - b)^+$ (also called *positive part of deviation* $\Sigma a_j x_j - b$), shortage denoted by $(\Sigma a_j x_j - b)^-$ (also called *negative part of deviation* $\Sigma a_j x_j - b$). Both are always ≥ 0.

If $\Sigma a_j x_j \geq b$, excess = $\Sigma a_j x_j - b$, Shortage = 0.

If $\Sigma a_j x_j \leq b$, excess = 0, Shortage = $-(\Sigma a_j x_j - b)$.

Minimizing linear combinations of such excesses and shortages with coefficients ≥ 0, can be transformed into LP.

Example: Transportation problem with plants P_1, P_2, markets M_1, M_2, M_3.

c_{ij} = Unit shipping cost from P_i to M_j

d_j, p_j, s_j = Demand, selling price (\$/unit) upto demand, selling price to warehouse when shipped quantity exceeds demand, at M_j.

a_i, b_i, g_i, h_i = Regular time production capacity, overtime production capacity, regular time production cost (\$/unit), overtime production cost, at P_i.

	c_{ij} for $j=$ 1	2	3	a_i	b_i	g_i	h_i
$i=1$	11	8	2	900	300	100	130
2	7	5	4	500	200	120	160
d_j	400	500	200				
p_j	150	140	135				
s_j	135	137	130				

Formulate the best production, shipping plan to maximize net profit.

Multiobjective LP Models

Several objective functions to optimize simultaneously.

Usually conflicts among objectives. Best solution for one, may be worst for another.

Efficient Point: Consider Minimizing $z_1(x), \ldots, z_k(x)$ simultaneously. A feasible solution $\bar{x}$ such that: there exists no feasible solution x satisfying all following inequalities, at least one strictly.

$$
\begin{aligned}
z_1(x) &\leq z_1(\bar{x}) \\
&\vdots \\
z_k(x) &\leq z_k(\bar{x})
\end{aligned}
$$

Usually, very large no. of efficient points. Algorithms exist for finding all, but not of practical use, because difficult to determine best among them.

Need: Approximate tradeoff information, i.e., how much value of one obj. func. can be sacrificed for unit gain in another. Or, priorities among objective functions.

Two commonly used practical techniques.

Positive Combination Technique: First put: either all objectives in maximization form, or all in minimization form. Define all of them in common units, say money units, or scores, etc.

From tradeoff or priority information, determine positive weights for various objective functions, $w_1, \ldots, w_k$, the higher the weight, the more important the objective function. Then optimize weighted combination $\Sigma_{t=1}^{k} w_t z_t(x)$, and take opt. sol. as the best sol. If all weights > 0, this will be an efficient point.

Example: Fertilizer problem. Maximize

$z_1(x) =$ Daily net profit $= 15x_1 + 10x_2$

$z_2(x) =$ Market share $= 300x_1 + 175x_2$

$z_3(x) =$ Hi-Tech Market share $= 300x_1$

Take weights to be 0.5, 0.25, 0.25.

GOAL PROGRAMMING APPROACH

Most Popular for multiobjective problems. Here objectives can be in different units, and some can be in min form, others in max form.

For each, select a *goal* or *target value*, i.e., a desirable vaue for it.

Objectives $\quad z_1(x), \ldots, z_k(x)$

Goals $\quad g_1, \ldots, g_k$.

$u_t^+ = (z_t(x) - g_t)^+, \quad u_t^- = (z_t(x) - g_t)^-$ are excess, shortage in tth objective.. They satisfy:

$$z_t(x) - g_t = u_t^+ - u_t^-, \quad \text{for } t = 1 \text{ to } k$$

Now construct a *PENALTY FUNCTION*, a nonnegative combination of these excesses and shortages, $\Sigma_{t=1}^k(\alpha_t u_t^+ + \beta_t u_t^-)$, where

α_t	> 0	if it is desirable to have $z_t(x)$ as small as possible.
	$= 0$	if it is desirable to have $z_t(x)$ as large as possible.
β_t	> 0	if it is desirable to have $z_t(x)$ as large as possible.
	$= 0$	if it is desirable to have $z_t(x)$ as small as possible.
both α_t, β_t	> 0	if it is desirable to have $z_t(x)$ equal to its goal g_t as far as possible.

The goal programming approach minimizes this penalty function, and its opt. sol. as the best sol.

Example: Consider fertilizer problem.

$z_1(x)$ = Net daily profit $= 15x_1 + 10x_2$, goal = \$13,000.

$z_2(x)$ = Market share = Daily tonnage sold $= x_1 + x_2$, goal = 1150 tons.

$z_3(x)$ = Hi-Tech Mkt. share = Daily Hi-Ph sold = x_1,   goal = 400 tons.

All objective functions to be maximized. Take penalty coeffs. for shortages from goals to be 0.5, 0.3, 0.2 respectively.