0.1

O. R.

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Introductory Lecture

Operations Research is the branch of science dealing with techniques for *optimizing* the performance of systems.

System is any organization, large or small.

Performance Measure Any criterion measuring the way system is working, that is considered important. For example, if system is a company, some important performance measures are: total yearly profit, market share, etc.

Optimizing A Performance Measure Performance measures are of two types:

PROFIT MEASURE: is one, for which the higher its value, the more desirable it is.

COST MEASURE: is one, for which the lower its value, the more desirable it is.

Optimizing a performance measure means maximizing it if it is a profit measure, or minimizing it if it is a cost measure.

Optimization involves Maximization or Minimization as desired.

To optimize implies Decision making to determine best operating conditions of system that yield best values for performance measures.

Main function of O. R. $\longrightarrow$ Tool for decision making.

Optimization is main pillar of O. R.

Everyone is interested in Optimization!

O. R. Techniques:

Multicriteria Decision-making : To chose the best among a small number of alternatives, when there are many important characteristics to consider.

Linear Programming: To optimize a single linear objective function subject to linear constraints. A technique with many applications.

Network Programming: Optimization problems that can be described over a graph structure. Many applications in transportation, distribution, etc.

Dynamic Programming: Technique for solving multi-period or multi-stage decision problems.

CPM for Project Scheduling: Techniques for controlling, scheduling various jobs in a project, which have to obey a precedence relationship.

Integer and Combinatorial Optimization: Techniques for solving mathematical models involving discrete decision variables.

Nonlinear Programming: Techniques for solving mathematical models involving nonlinear functions.

Mathematical Programming: Includes linear, network, integer, combinatorial, and nonlinear programming.

Multi-objective Programming: Techniques for obtaining good solutions to optimization models involving two or more objective functions that need to be optimized simultaneously. Goal Programming is one technique for solving multi-objective programs.

Inventory Management: Techniques for maintaining inventories to meet demands for goods, while at same time minimizing inventory holding costs.

Queing Problems: To optimize performance of systems involving waiting lines.

Simulation: Techniques for modeling the operations of a system through computer simulation.

Algorithms

Many of the problems in O. R. are solved using procedures called algorithms.

An Algorithm for a Problem: is a systematic step by step procedure for solving the problem.

History: Word comes from title of Latin translation

Algoritmi de Numero Indorum

meaning *Al-Khwarzmi Concerning the Hindu Art of Reckoning* written in 825 AD by Arabic Scholar Muhammad Ibn Musa Al-Khwarizmi who lived in Iraq, Iran (mostly Bagdad), based on earlier Indian and Arabic treatizes.

Algorithms seem to have originated in work of ancient Indian mathematicians on rules for solving linear and quadratic equations.

An Example: Problem: Input: Given Positive Integer n. Find all primes $\leq n$.

Eratosthenes Sieve (3rd century BC)

Step 1: Write all integers 1 to n. Strike off 1 and set a pointer at 2, the first prime.

General Step: Suppose pointer now on r. Counting from r, strike off every rth number to right.

If all numbers to right of r now struck off, all numbers not struck off are all primes $\leq n$, terminate.

Otherwise move pointer right to next number not struck off , and repeat this general step with pointer in its new position.

EXERCISE: $n = 15$.

Classification of Optimization Problems

Optimization Problem: To select best among available alternatives

Category 1: Number of available alternatives very small: Usually solved by a *Scoring Method*.

Category 2: No. of alternatives finite but large: Integer & combinatorial optimization problems belong to this category.

Category 3: Number of alternatives ∞: Continuous variable problems (linear and nonlinear optimization problems) belong to this category.

For category 2, 3 problems, we need to construct a Mathematical Model for the problem, and solve it using an efficient algorithm.

Decision Making by Quantitative Analysis

For problems which are very important, or very complex, good solutions cannot be developed without the aid of quantitative analysis.

Quantitative analysis requires the representation of the problem using a mathematical model. Mathematical modeling is a critical part of the quantitative approach to decision making. Constructing mathematical models, to represent real world problems reasonably closely, is an essential skill for all engineers.

Two types of factors effecting system performance:

- **Uncontrollable factors:** Factors such as environmental factors, not under the control of the decision maker.
- **Controllable inputs:** Factors whose values can be controlled by the decision maker, which affect the functioning of the system. These factors are called the *decision variables* in the model.

Deterministic models: If all uncontrollable inputs are known exactly and do not vary, system performance depends only on values of decision variables, model becomes a deterministic model. No uncertaity.

Stochastic or Probabilistic models: When uncontrollable inputs are uncertain and subject to variation. Here system performance uncertain even when the values of all decision variables are fixed.

We study deterministic models in this course.

Steps in Decision Making by Quantitative Analysis:

Model Building : Build a mathematical model that represents how system functions (identify decision variables, constraints on them, and objective function that measures system performance).

Solve Model : Use an efficient algorithm to get an optimum solution.

Implement solution, or Update Model & Repeat : Check solution for practical feasibility. If it is not, make necessary changes in model & repeat.

Also, many times decision makers use their practical knowledge to transform optimum solution into an implementable solution.

Early History of Related Topics

Linear Algebra = Branch of Math. dealing with Systems of linear eqs. (Note: No inequalities!).

Example: Steel company has four scrap metals (SM-1 to SM-4) with following compositions.

Type	% in type, by weight, of element			
	Al	Si	C	Fe
SM-1	5	3	4	88
SM-2	7	6	5	82
SM-3	2	1	3	94
SM-4	1	2	1	96

Need to blend into a mixture with composition by weight : Al = 4.43%, Si = 3.22%, C = 3.89%, Fe = 88.46%. Determine how to prepare this mixture.

Elimination Method for linear eqs. developed by 3rd century BC by Chinese & Indian mathematicians.

Unknown in Europe until C F Gauss discovered it (19th century) while studying linear eqs. to approximate orbit of asteroid Ceres using **Method of least squares**. W. Jordan discussed this method in a book he wrote (1888). Hence method called GJ Elimination method. (A better version called G Elimination method).

Both GJ & G methods cannot directly handle inequalities.

Linear Programming LP, Methods for handling systems of linear eqs. & ineqs. Started with:

Farka's Lemma: Fundamental theoretical result on linear ineqs. (1901)

G B Dantzig's **Simplex Method** to solve LPs (1947).

So, LP, a 20th century topic.

All methods for LP are iterative methods requiring sol. of 2 systems of linear eqs. in each iteration.

GJ Method for Solving Linear Eqs.

Consider system

$$(1) \qquad Ax = b$$

where **coefficient matrix** A is $m \times n$ matrix, & RHS vector $b \in R^m$.

Matrix $A = (a_{ij})_{m \times n}$: Started as a convenient way to represent the coefficients in a system of linear equations. Matrices and matrix algebra introduced by J J Sylvester, and Cayley in 1840s.

$A_{i.} = i$th row vector of the matrix A, $i = 1$ to m.

$A_{.j} = j$th column vector of the matrix A, $j = 1$ to n.

Unit or Identity Matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Permutation Matrix; One whose rows can be rearranged to become the unit matrix.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$A_{m\times n}$. Products

- Ax defined only if x is a column vector in R^n.
- πA defined only if π is a row vector in R^m.

Main Computational tools

Row Operations:

1. Multiply a row by a nonzero scalar.
2. Add a scalar multiple of a row to another.
3. Interchange two rows.

Example:

$A_{1.}$	1	0	3	-5
$A_{2.}$	0	1	2	4
$(1/2)A_{1.}$				
$A_{2.} - 5A_{1.}$				

Gauss-Jordan Pivot Operation or Pivot Step: Specified by specifying the *pivot column* and *pivot row*.

IMPORTANT CONDITION: *Pivot Element*, element in the pivot row and the pivot column, should be nonzero.

The GJ Pivot operation consists of a series of row operations to convert pivot column into *unit column* with 1 entry in pivot row.

Example: Here PC = pivot column, PR = pivot row.

$$
\begin{array}{rrrrr|rl}
 & & \text{PC} & & & & \\
\hline
1 & 0 & 1 & -1 & 2 & 1 & \\
6 & 4 & \boxed{2} & -4 & 0 & 6 & \text{PR} \\
3 & -2 & 0 & 1 & -1 & 4 & \\
1 & 0 & -1 & 3 & 4 & 10 & \\
\hline
 & & & & & & \\
\hline
\end{array}
$$

History: Developed as a tool for solving systems of linear equations by the *Elimination method*

Example:

$$
\begin{array}{rrr|r}
x_1 & x_2 & x_3 & \\
\hline
2 & 2 & 2 & 12 \\
-1 & 0 & 1 & 1 \\
1 & -1 & 0 & -2 \\
\hline
\end{array}
$$

Constraints in (1) correspond to rows of coefficient matrix A. System (1) in constraint representation is:

$$(1) \qquad A_{i.}x = b_i,\ i = 1 \text{ to } m.$$

Multiply ith constraint by π_i, $i = 1$ to m, and add. This leads to linear combination

$$(2) \qquad (\Sigma_{i=1}^m \pi_i A_{i.})x = \Sigma_{i=1}^m \pi_i b_i.$$

If

$$(3) \qquad \Sigma_{i=1}^m \pi_i A_{i.} = 0, \text{ the 0-row vector}$$
$$\Sigma_{i=1}^m \pi_i b_i = 1$$

then (2) is the **inconsistent eq. $\mathbf{0 = 1}$**. So, if $\pi = (\pi_1, \ldots, \pi_m)$ satisfying (3) exists, (1) has no sol. x. In matrix notation

$$(3) \qquad \pi A = 0, \quad \pi b = 1$$

Theorem: System (1) has no solution x iff (3) has a solution π.

Both these systems (1), (3) can be processed simultaneously by the Gauss-Jordan (GJ) method applied on system (1). For this, it is convenient to record (1) in the form of a detached coefficient tableau. The GJ method tries to carry out a GJ pivot step in each row of this tableau with the aim of creating a unit submatrix of order m on its left hand side. In this process, at each stage, each row in the current tableau will always be a linear combination of rows in (1). For each row vector in the current tableau, the coefficients in this linear combination will be denoted by a row vector $\mu = (\mu_1, \ldots, \mu_m)$. These μ-vectors for the various rows in the current tableau are stored and updated under a "**memory matrix**". These μ-vectors for the various rows in the original tableau are the unit vectors in R^m. These are recorded in the original tableau before beginning the application of the GJ method.

Memory matrix*				Original tableau				
π_1	π_2	$\dots$	π_m	x_1	x_2	$\dots$	x_n	
1	0	$\dots$	0	a_{11}	a_{12}	$\dots$	a_{1n}	b_1
0	1	$\dots$	0	a_{21}	a_{22}	$\dots$	a_{2n}	b_2
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	0	$\dots$	1	a_{m1}	a_{m2}	$\dots$	a_{mn}	b_m

*Coeff. vector for expressing the row on the right of this memory matrix as a linear combination of rows in the original tableau.

Carrying out all the computations involved in the pivot steps, also on the columns of the memory matrix, updates it. Here is a summary of the method.

1. Select the order in which rows 1 to m in the tableau are to be chosen as pivot rows.

2. **General Step:** Suppose row r is the pivot row for the pivot step in the present tableau. Let $\bar{a}_{r1}, \dots, \bar{a}_{rn}, \bar{b}_r$ be the coefficients of the variables and the updated RHS constant in row r in the present tableau.

2.1. If $(\bar{a}_{r1}, \ldots, \bar{a}_{rn}) \neq 0$ select a variable x_j for a j such that $\bar{a}_{rj} \neq 0$ as the basic variable in row r, and the column of x_j in the present tableau as the pivot column, and perform the GJ pivot step. If row r is the last pivot row in the selected order, go to 3 if 2.3 given below has never occurred so far. If row r is not the last pivot row in the selected order, with the resulting tableau go back to 2 to perform a pivot step with the next pivot row in the selected order.

2.2. If $(\bar{a}_{r1}, \ldots, \bar{a}_{rn}) = 0$ and $\bar{b}_r = 0$, this row is called the "0 = 0" equation. This indicates that the constraint in the original system (1) corresponding to this row is a redundant constraint and can be eliminated without changing the set of solutions.

If row r is the last pivot row in the selected order, go to 3 if 2.3 given below has never occurred so far. If row r is not the last pivot row in the selected order, with the present tableau go back to 2 to perform a pivot step with the next pivot row in the selected order.

2.3. If $(\bar{a}_{r1}, \ldots, \bar{a}_{rn}) = 0$ and $\bar{b}_r \neq 0$, this row is called the "$0 = \alpha$" equation for $\alpha = \bar{b}_r \neq 0$, or an inconsistent or infeasible equation. In this case the original system (1) has no solution.

If $(\bar{\mu}_1, \ldots, \bar{\mu}_m)$ is the row in the memory matrix in row r in the present tableau, then $\bar{\pi} = (1/\bar{b}_r)(\bar{\mu}_1, \ldots, \bar{\mu}_m)$ is a solution of the alternate system (3).

If it is only required to either find a solution to (1) or determine that it is inconsistent, the method can terminate here.

3. Make all the nonbasic variables in the final tableau equal to 0, and the basic variable in each row equal to the updated RHS constant in that row in the final tableau. This is a basic solution to (1), terminate.

EXAMPLE 1:

	x_1	x_2	x_3	x_4	x_5	RHS
	-1	0	$\boxed{1}$	1	2	10
	1	-1	2	-1	1	5
	1	-2	5	-1	4	20
	2	0	0	-2	1	25
	3	-2	5	-3	5	45
	5	-2	5	-5	6	70

EXAMPLE 2:

	x_1	x_2	x_3	RHS
	1	1	$\boxed{1}$	5
	-1	3	1	6
	0	4	2	11
	2	3	1	10

EXAMPLE 3:

	x_1	x_2	x_3	x_4	x_5	RHS
	2	1	-1	-2	0	4
	4	1	5	6	4	9
	6	2	4	4	4	12
	10	3	9	10	8	22