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Quantitative flow visualization using the hydraulic analogy

S. L. Rani, M. S. Wooldridge

Abstract The current work describes the development of a non-intrusive optical method for the quantitative determination of water heights along a hydraulic jump in shooting water flows on a water table. The technique involves optically superimposing a series of alternating dark and clear fringes on the water flow. It is proposed that the fringe deviations seen under a hydraulic jump can be simulated using a series of optical prisms oriented along the direction of the hydraulic jump. The height of each prism gives the local maximum water height at the fringe location. Three types of theoretical prism configurations (isosceles flat-topped prism, scalene flat-topped prism and rounded-topped prism models) have been studied for two flow systems: shooting flow around a wedge and around a cylinder. Equations relating the physical characteristics of the deviated fringes to the height of the theoretical prism and hence the local water height are presented. The variation in water height along a hydraulic jump for flow around a wedge obtained using the optical technique has been compared with heights obtained using a depth gauge. The results were in good agreement for the range of Froude numbers studied ($Fr = 1.9 - 3.6$). The rounded-topped prism model led to the best agreement with the physical measurements, within 11% throughout the range of conditions studied. The uncertainty associated with the water height determination using the optical technique is $\pm 10\%$.

Nomenclature

Roman

f	Intermediate variable (used in mathematical derivation)
h	Water height
l	Length of the fringe deviation
Fr	Froude number
H	Prism height (also height of the local peak of the hydraulic jump)

Greek

α	Hydraulic jump angle (also prism orientation angle)
ϕ	Wedge angle
κ	Ratio of specific heats
μ	Refractive index of the prism material
θ	Prism apex angle
ψ	Fringe deviation angle

Subscripts

1, 2	Asymmetric prism properties
ref	Reference condition
no subscript	Local variable

1
Introduction

The analogy between pressure waves in a compressible gas and long gravity waves on the free surface of a shallow liquid is well known (Shapiro 1954). Qualitative evidence of the analogy can be readily obtained by comparing the wave patterns observed in supersonic wind tunnels with the waves seen on the surface of a shallow sheet of water flow. The quantitative aspects of the analogy are deduced based on the analogous equations of continuity and energy conservation for two-dimensional, isentropic flow of a gas having a ratio of specific heats $\kappa = 2.0$, and quasi-two-dimensional frictionless flow of an incompressible liquid (Shapiro 1954). Using these relations, the local liquid height h can be related to the local gas density, temperature and pressure.

The primary application of the hydraulic analogy is in the study of supersonic gas flows. Supersonic flow of a gas is analogous to shooting flow of a liquid (Black and Mediratta 1951). A shock wave is formed when supersonic gas flow encounters a body such as a wedge. The analogous entity in the case of shooting water flow is the hydraulic jump.

The water table is the primary test facility used for studying the hydraulic analogy. Because the water heights on the table

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correspond to the desired gas properties, it is important to measure the heights in the flow field accurately. A contact-type gauge or probe is intrusive, and hence disturbs the flow by introducing capillary waves at the point of contact with the water surface, leading to uncertainties in the height measurements. In addition, it is difficult to simultaneously measure water heights at multiple locations in the flow field using depth probes. Consequently, there is a need for a non-intrusive, yet quantitative technique for simultaneously measuring flow heights at a large number of points in water table studies.

The current work proposes a new non-intrusive, optical technique for measuring flow heights along a hydraulic jump in a water table. The work is based on a technique developed by Pal and Bose (1993, 1994), where the authors proposed that the deviation of a mechanical fringe when viewed through a hydraulic jump (see Fig. 1) can be simulated by a rounded-top prism oriented along the direction of the hydraulic jump. Pal and Bose suggested that under these conditions the height of the prism corresponds to the local flow height. The technique was empirically calibrated for a range of prism configurations (i.e. flow conditions). However, the experimental results for pressure distribution for shooting flow around a wedge were consistently higher than those determined by numerical simulation (greater than 20% throughout the flow field).

In the present work, explicit equations are derived relating the height of the hydraulic jump to the physical characteristics of the fringe deviations via different theoretical prism models. Several prism geometries are proposed and evaluated including rounded-topped prism, isosceles flat-topped prism and scalene flat-topped prism models. A digital image acquisition system is used to further improve the measurement technique by eliminating the need for film processing and developing.

2
Theory (optical ray-tracing)

The proposed technique is based on the similarities between the distortions of fringes produced by the changes in water height in shallow water flow and the distortions of fringes produced by optical prisms. In a water table study, alternating dark and clear fringes are optically superimposed on the flow. When shooting water flow ($Fr > 1$) encounters the model, a hydraulic jump is produced. When viewed through a hydraulic jump, the fringes of the mechanical grating appear deviated. It has been found that the length (l) and the angle (ψ) of the fringe deviation (see Fig. 1), can be related to the local maximum water height at that fringe location. The hypothesis at the basis of the measurement technique is that a hydraulic jump can be approximated by a series of prisms oriented along the direction of the hydraulic jump. The height of the prism at each fringe location along the hydraulic jump, H, is equal to the local maximum water height, h. In other words, the height of the prism is equal to the height of the water at the apex of the hydraulic jump ($H = h$).

The angle of fringe deviation seen under a prism is related to the apex angle, θ, the angle of orientation, α, and other physical characteristics of the prism which depend on the type of prism cross-section used. Figure 2 shows a schematic of the system for an isosceles flat-topped prism configuration. The relationship between ψ, θ and α can be obtained by ray tracing through the prism. The derivation is straightforward and results in a relationship for the height of the prism as a function of the length of the fringe deviation, the angle of the fringe deviation, the angle of the prism orientation and the apex angle of the prism, i.e. $H = f(\alpha, l, \psi, \theta)$. Details of the derivations for the isosceles flat-topped prism model and the other prism models are provided in Rani (1998).

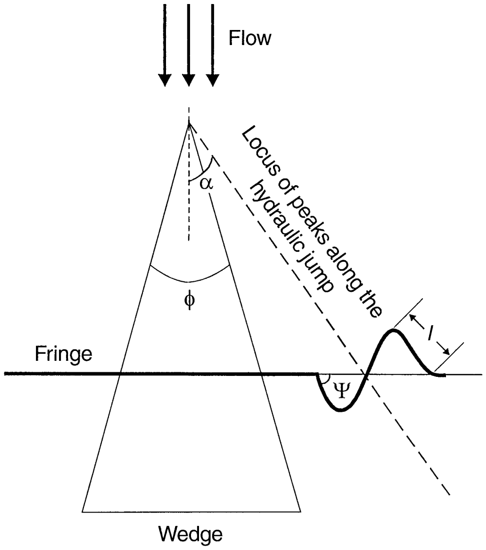

Fig. 1. Schematic of fringe deviation as viewed through a hydraulic jump (after Pal and Bose, 1993)

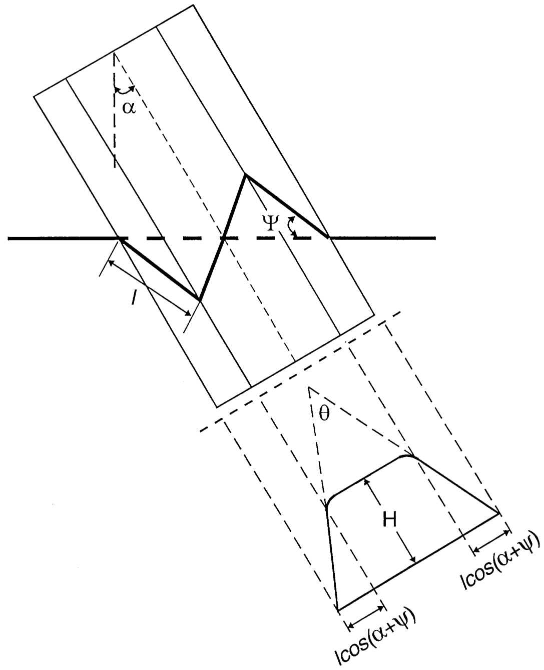

Fig. 2. Top and front views of the theoretical isosceles flat-topped prism indicating the angle of orientation α, the deviated fringe properties l and ψ and the prism height H

The angle of deviation of the fringe, ψ, can be determined from the system shown in Fig. 2 and is given by the following expression:

$$\psi = \sin^{-1}\left(f\sin\alpha/\sqrt{f^2+\sec^2\alpha-2f}\right) \quad (1)$$

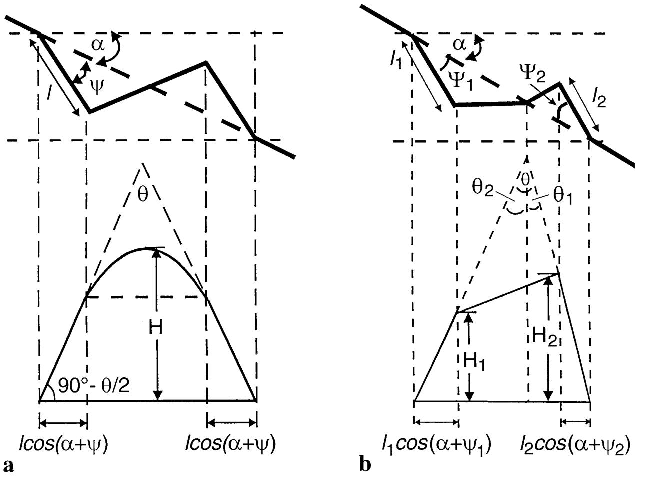

Fig. 3a,b. Schematics representing the rounded-topped prism model **a** and the scalene flat-topped prism model **b** geometries. The figures are not to scale and the deviations in **b** have been exaggerated to show the asymmetry

where

$$f=\frac{\cot(\theta/2)}{\cot(\theta/2)+\tan\left((\theta/2)+\sin^{-1}\left(\frac{\cos(\theta/2)}{\mu}\right)\right)} \tag{2}$$

and μ is the refractive index of the prism material.

The length of the fringe deviation, as seen under the flat-topped isosceles prism is directly related to the length of the lateral side of the prism (see Fig. 2). Using ψ, θ and l for the flat-topped isosceles prism, the height H of the prism, and hence the height of the local hydraulic jump peak, can be obtained using Eq. (3):

$$H=l\cos(\psi+\alpha)\cot(\theta/2) \tag{3}$$

A rounded-top prism with a parabolic curve-fit at the truncated portion was also investigated and is shown schematically in Fig. 3a.

One assumption made in deriving Eq. (1) is that the fringe deviations under a hydraulic jump can be simulated by a series of isosceles or symmetric prisms. In other words, we are assuming that the fringe deviation angles on either side of a hydraulic jump are equal (see Figs. 2 and 3a). However, the actual fringe deviations observed in water table experiments are not exactly symmetric. The asymmetry can be accounted for by means of relations derived for a scalene prism, where the length of the fringe deviations are not the same on each side of the prism. The local water height, H, in the case of a scalene prism was assumed to be the arithmetic mean of the two heights, H_1 and H_2 (see Fig. 3b).

3
Experimental apparatus

Experiments were conducted at the Emil Buehler Aerodynamic Analog Laboratory at Texas A&M University to evaluate the proposed optical diagnostic. Figure 4 is a schematic of the top view of the water table. The bed and the side walls of the table are made of clear glass to minimize friction and to allow for flow visualization. To reduce turbulence, water enters the supply tank through a fine wire mesh. In addition, flow straighteners within the supply tank help reduce surging and other non-uniformities in the supply flow.

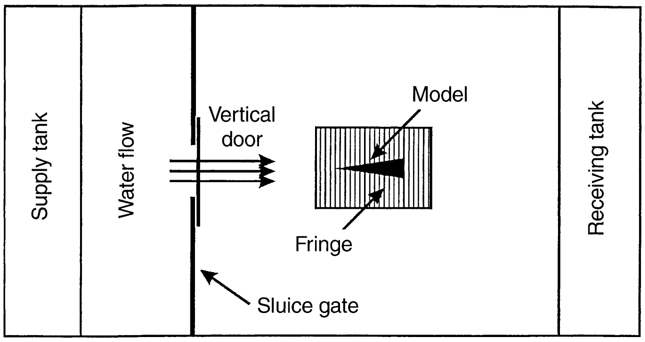

Fig. 4. Schematic of water table facility showing flow configuration

A mechanical grating consisting of a series of alternating dark and clear fringes was placed on the bottom surface of the water table. The area of the water table encompassed by the grating constituted the test region. The dark fringes were 0.51 mm thick and were separated by an edge-to-edge distance of 7.62 mm.

The pumping power was not sufficient to maintain shooting water flow ($Fr > 1$) across the entire cross-section of the water table. Hence, a sluice gate was installed across the water table to reduce the cross-section. The slit height could be varied by means of a vertical gate. The water from the supply tank was accelerated to Froude numbers >1 as it exited the sluice gate through the rectangular slit. The height of the slit was adjusted such that an undisturbed (without the model present) optimum water height of approximately 6–7 mm (Shapiro 1954) was maintained in the test region. In the current experiments, the height of the water reservoir upstream of the sluice gate (with respect to the glass floor of the water table) was taken as the reference height (h_{ref}). The sluice gate also served to ensure a smooth and steady flow in the test region.

For comparison purposes, depth gauge measurements were also made at discrete locations along the hydraulic jump. The water height measuring apparatus consisted of a needle probe attached to a vernier caliper (accuracy of ± 0.01 mm). The caliper was attached to a traversing platform mounted vertically above the water table. To measure the water height at a point, the caliper was zeroed when the probe tip touched the surface of the table ($h=0$). The probe was then raised until the tip cleared the water surface and was then lowered until it just touched the water surface.

Another important component of the experimental apparatus was the image capturing and processing system. A light source was used to illuminate a flat, white reflecting surface. The surface directed the light vertically through the transparent floor of the water table such that the model and fringes were lit with uniform intensity. A video camera was mounted on the traversing platform above the test region and was used to obtain images of the fringes. Analog images from the camera were converted into digital images using video capture hardware and software. Image analysis was then conducted on a personal computer.

4
Results

The water heights along the hydraulic jump were determined by measuring α, ψ and l from the digital images. Once ψ and α were obtained for each fringe along the hydraulic jump, the corresponding values of θ were obtained via Eqs. (1) and (2). Using the ψ, α, θ and l values, the heights H were determined using the appropriate relations as specified by the choice of prism model. Three prism models were examined: an isosceles flat-topped prism, a scalene flat-topped prism and a rounded-topped prism model.

Two flow systems were studied: flow around a wedge (with a wedge angle of $\phi=20°$) and flow around a cylinder (26 mm diameter). The wedge was located on the water table at a zero angle of attack. Images were taken over a range of Froude numbers (1.9–3.6). For flow around the wedge, heights along the hydraulic jump were obtained using the three prism models and the depth probe. Depth-probe data were not obtained for the cylinder experiments for reasons discussed below. Figure 5 shows a typical image obtained for flow around the wedge model. Note that since the fringe widths are approximately constant throughout the deviation from horizontal (as seen in Fig. 5), refraction effects were considered negligible.

In Fig. 6, the water heights along the hydraulic jump predicted by the isosceles flat-topped prism, the scalene flat-topped prism and the rounded-topped prism models are compared with water heights measured using the depth-probe for a condition of $Fr=2.27$. Measurements were difficult to make near the nose of the wedge, and hence the results are presented for regions where the fringe deviations were uniform and easy to identify ($x/c \gtrsim 0.2$ for $Fr=2.27$). The rounded-topped prism model slightly over predicts the water heights, however by less than 11% throughout the range of conditions studied. The isosceles flat-topped and scalene flat-topped prism models both under predict the water heights, by less than 27% and 20%, respectively. The results were similar for the range of Fr numbers studied therefore only the results for $Fr=2.27$ are presented. The uncertainty associated with the water height measurement using the optical technique is small ±10%, and is primarily due to uncertainties in the physical measurements of deviated fringe parameters.

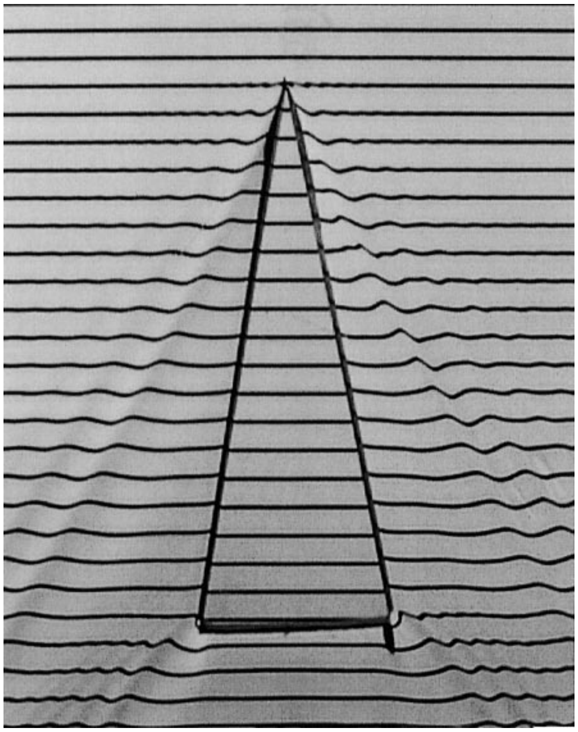

Fig. 5. Typical image of shooting flow around a wedge at zero angle of attack, $Fr=3.6$ and $\phi=20°$

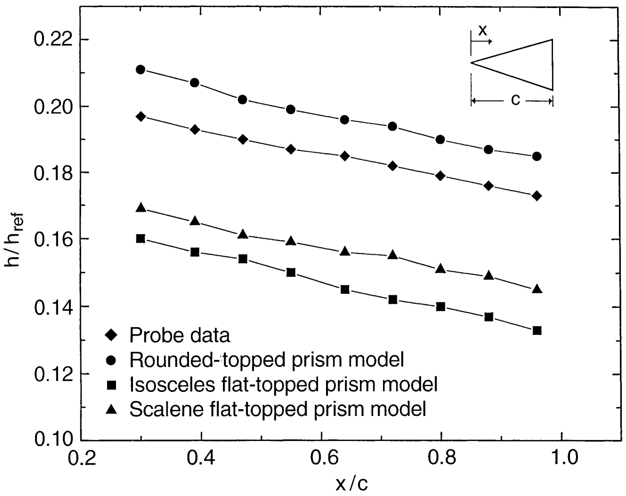

Fig. 6. Comparison of water height data obtained optically with water height data obtained using a depth probe ($Fr=2.27$ and $\phi=20°$). The reference height is $h_{\mathrm{ref}}=32$ mm, and x/c is the dimensionless distance (relative to the chord length of the model, $c=10.2$ mm).

An explicit comparison with depth gauge measurements is not possible for the cylinder flow system due to unsteady flow conditions caused by the model. However, the fringe deviations were clearly discernible and reasonable values for water heights were obtained. The application of the deviated-fringe measurement technique to the cylinder flow system successfully demonstrated the application of the diagnostic to transient systems where other intrusive measurement techniques, such as the depth gauge, are not feasible. The hydraulic analogy would only apply to the continuity relations (or gas density and water height) for these systems.

5
Conclusions

The results obtained using the deviated-fringe optical technique and a rounded-topped prism model compared well with depth-probe measurements, within ±11% throughout the range of conditions studied. The isosceles flat-topped prism model significantly under predicted the water heights. The scalene flat-topped prism model attempts to address asymmetry found in the experimental images. The scalene flat-topped prism model also under predicted the water heights, however the results were improved in comparison with the isosceles flat-topped prism model.

The water table continues to provide an economical alternative to supersonic wind tunnel experiments. The deviated-fringe measurement technique presented in the current work is similarly inexpensive to implement. The optically based technique allows multiple non-intrusive measurements to be obtained simultaneously at a high level of accuracy. The derivation of explicit relations relating water height to the physical characteristics of the deviated fringes eliminates the need for elaborate calibration procedures. By employing rapid digital image acquisition hardware and software, the deviated-fringe technique can be readily extended to transient water table studies. The deviated-fringe optical diagnostic technique is quantitative, non-intrusive and accurate (uncertainty ±10%), meeting a fundamental need in the water table community.

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