Homework Set 8

Please, try to do all of the following problems. Solutions to three of them are due on Monday March 26.

Problem 1. Let k be an algebraically closed field and $X = \mathbb{P}^n_k$, that we identify with the set of lines in $\mathbb{A}^{n+1}$.

(1) Show that the set
$$Z = \{(P, \text{Ł}) \in \mathbb{A}^{n+1} \times \mathbb{P}^n \mid P \in L\}$$
is a closed subset of $\mathbb{A}^{n+1} \times \mathbb{P}^n$.
(2) Show that if we consider Z with the reduced subscheme structure, then the projection onto the second factor induces a morphism $\pi\colon Z \to \mathbb{P}^n$ such that Z has a natural structure of line bundle over $\mathbb{P}^n$.
(3) Show that the sheaf of sections of π is isomorphic to $\mathcal{O}(-1)$.

Problem 2. Let $f\colon X \to Y$ be a morphism of schemes.

(1) Show that the pull-back induces a group homomorphism
$$f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X).$$
(2) Suppose that f is a dominant morphism of integral schemes. Show that for every Cartier divisor D on Y we can define a Cartier divisor $f^*(D)$ on X and in this way we get a morphism of groups $\operatorname{Cart}(Y) \to \operatorname{Cart}(X)$. Moreover, show that $\mathcal{O}(f^*(D)) \simeq f^*\mathcal{O}(D)$.

Problem 3. Give an example of a Weil divisor on a normal integral Noetherian scheme that is not Cartier.

Problem 4. Let X be a normal variety over an algebraically closed field k. Show that the map taking V to $V \times \mathbb{A}^n$ induces isomorphisms $\operatorname{Cl}(X) \simeq \operatorname{Cl}(X \times \mathbb{A}^n)$ and $\operatorname{Pic}(X) \simeq \operatorname{Pic}(X \times \mathbb{A}^n)$.

Problem 5. Show that if X is a normal variety over an algebraically closed field, then for every $n \geq 1$ we have isomorphisms $\operatorname{Cl}(X \times \mathbb{P}^n) \simeq \operatorname{Cl}(X) \times \mathbb{Z}$ and $\operatorname{Pic}(X \times \mathbb{P}^n) \simeq \operatorname{Pic}(X) \times \mathbb{Z}$.