Homework Set 7

Please, try to do all of the following problems. Solutions to three of them are due Monday March 19.

Problem 1. Let A be a Noetherian ring and $X = \mathbb{P}^n_A$. Show that if

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \ldots \to \mathcal{F}_p$$

is a complex of coherent sheaves on X and if $m \gg 0$, then the induced complex

$$\Gamma(X, \mathcal{F}_1 \otimes \mathcal{O}(m)) \to \Gamma(X, \mathcal{F}_2 \otimes \mathcal{O}(m)) \to \ldots \to \Gamma(X, \mathcal{F}_p \otimes \mathcal{O}(m))$$

is also exact.

Problem 2. Let $X = \mathbb{P}^n_k$, where k is a field. Show that for every coherent sheaf $\mathcal{F}$ on X, there is an exact sequence

$$0 \to \mathcal{E}_{n+1} \to \mathcal{E}_n \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

with all $\mathcal{E}_i$ locally free of finite rank. In fact, show that one can take all $\mathcal{E}_i$ to be direct sums of invertible sheaves.

Problem 3. Let k be a field and $X = \mathbb{P}^n_k$.

i) Show that if I is a homogeneous ideal of $S = k[x_0, \ldots, x_n]$, then the sheaf of ideals $\widetilde{I}$ defines a closed subscheme of X that is isomorphic to $\mathrm{Proj}(S/I)$.
ii) Show that two different ideals can give rise to the same closed subscheme.
iii) For every closed subscheme Y of X, there is a unique ideal I that is maximal with respect with inclusion and such that $\widetilde{I} = \mathcal{I}_Y$, the ideal defining Y. It is characterized by the fact that it is *saturated*, i.e. if $u \in S$ is such that $ux_i \in I$ for every i, then u is in I. Moreover, we have $I = \oplus_m \Gamma(X, \mathcal{I}_Y \otimes \mathcal{O}(m))$.

Problem 4. Let k be a field and $Y \subseteq \mathbb{P}^n_k$ a closed subscheme. The *coordinate ring* $S(Y)$ of Y is $k[x_0, \ldots, x_n]/I_Y$, where I_Y is the saturated ideal corresponding to Y. Consider also the ring

$$S'(Y) := \oplus_{m \in \mathbb{N}} \Gamma(\mathbb{P}^n_k, \mathcal{O}_Y \otimes \mathcal{O}(m)).$$

Show that there is an inclusion $S(Y) \subseteq S'(Y)$ that is an equality in large degree. Deduce that $S'(Y)$ is finitely generated as a k-algebra.

Problem 5. Let k be a field and $X = \mathbb{P}^n_k$.

i) Every radical ideal I in $S = k[x_0, \ldots, x_n]$ is saturated. Moreover, a closed subscheme Y of X is reduced if and only if the corresponding saturated ideal I_Y of S is radical.
ii) A closed subscheme Y of X is integral if and only if I_Y is a prime ideal.